CENTER OF PLANNING AND ECONOMIC RESEARCH

LECTURE SERIES

24.

OPTIMIZATION PROBLEMS IN PLANNING THEORY

By

HELMUT REICHARDT

Ruhr-Universität Bochum

ATHENS 1971

OPTIMIZATION PROBLEMS IN PLANNING THEORY

CENTER OF PLANNING AND ECONOMIC RESEARCH

LECTURE SERIES

24.

OPTIMIZATION PROBLEMS IN PLANNING THEORY

By

HELMUT REICHARDT

Ruhr-Universität Bochum

ATHENS 1971

Opinions or value judgements in this book are those of the author and do not necessarily represent those of the Center of Planning and Economic Research.

THE CENTER OF PLANNING AND ECONOMIC RESEARCH

The Center of Planning and Economic Research (KEPE) was founded in 1961 as an autonomous Public Organisation, under the title «Center of Economic Research», its basic objective being research into the problem of the operation, structure and development of the Greek economy. Another of its objectives was the training of young Greek economists in modern methods of economic analysis and research. For the establishment and operation of the Center considerable financial aid was provided by the Ford and Rockefeller foundations, and the United States Mission to Greece.

During 1964, the Center of Economic Research was reorganised into its present form, as the Center of Planning and Economic Research. In addition to its function as a Research and Training Institute, the Center, in its new form, was assigned the following tasks by the State: (1) The preparation of draft economic development plans, (2) the evaluation of public investment programmes and, (3) the study of short-term development in the Greek economy and advising on current problems of economic policy.

For the realisation of these aims, the KEPE, during its first years of operation (1961-66) collaborated with the University of California at Berkeley. The latter helped in the selection of foreign economists who joined the Center, to carry out scientific research into the problems of the Greek economy and in the organisation of an exchange programme, including visits of American *students to the Center, and the post-graduate training of young Greek economists at American universities.*

The research activity of the KEPE into the problems of the Greek economy, is presented in two series of publications, the «Research Monograph Series» and the «Special Studies Series A and B». The «Research Monograph Series» includes studies which, in addition to their practical interest, also have a theoretical interest. The «Special Studies Series A and B» mainly include studies of an empirical content. More specifically, Series A includes studies referring to fundamental problems of economic and social conditions in Greece and is distinguished from Series B by the fact that it includes a more systematic and detailed analysis of the subjects covered.

The Center has also developed a broad programme of scholarships or post-graduate studies in economics. Thus, in collaboration with foreign universities and international organisations, a number of young economists from Greece are sent abroad each year, to specialize in the various fields of economics. In addition, the KEPE organises a series of training seminars and lectures, frequently given by distinguished foreign scholars invited for that purpose to Greece. The lectures presented at these seminars are published in two series under the title: «Training Seminar Series» and «Lecture Series».

In addition to the above, the KEPE maintains contact with similar institutions abroad, and exchanges publications concerning development in methods of economic research, thus contributing to the promotion of the science of economics in the country.

TABLE OF CONTENTS

OPTIMIZATION PROBLEMS IN PLANNING THEORY

1. INTRODUCTION

In the context of the policy of economic development, the possibility of realising development plans has to be judged, more or less optimal development strategies have to be outlined or a past period of development has to be considered critically. The possibilities of realising quantitative aims of development plans vary considerably in the different countries. But there are countries showing especially favourable conditions for the realisation of such aims. These are the qualifications of the economic decision-makers and the administration, the economic discipline of the population and, last but not least, the quality and availability of statistical material. For such countries, economic models can be constructed as a basis for optimal plans, giving an answer to the problems stated above.

The models considered here are such for the first steps on the way towards these aims, and should be understood as preliminary studies. The purpose of these models is to discuss the special problems appearing in these kinds of models.

Each economic growth process, especially each development process, is characterized by a corresponding process of capital accumulation. Therefore, the following problem arises: How is the available net product to be allocated? In the relevant literature the following criteria are mostly discussed: The economic plan is to be realised as rapidly as possible, or the plan is to be realised in such a manner that total consumption is a maximum during the planning period. Instead of maximizing total consumption, sometimes the maximization of general utility functions is considered.[1] Therefore, the problem of an optimal planning can be formulated: A process which can run in alternative ways is to be controlled in such a manner that an objective, related to a welfare criterion, is maximized. Certain models of this kind are considered in the following.

---

1. Chakravarty, Goodwin, Ramsey, Samuelson, Tinbergen [1]

2. DYNAMIC SYSTEMS

The state of an economic system can be described by magnitudes, which can be considered as realisations of variables. These variables describe well-defined, interesting economic items, as for example the production of a sector, the capital stock or the consumption of an economy, the size of the population, the labour-input and so on. Such magnitudes are always related to time points or time intervals. Each date is described by a distinct time point, i.e. a distinct value of the time coordinate t. Two cases are distinguished: All points of a time interval are considered (continuous case) or only distinct time points of a time interval are considered (discrete case)

We especially speak of dynamic systems,[1] if variables which are related to a distinct time point are functions of variables related to the same or earlier time points. In the discrete case, a dynamic system can be illustrated by Tinbergen's[2] arrow scheme. The variables and the relations of the dynamic system are called dynamic models. We distinguish between continuous and discrete dynamic models.

---

1. Frisch
2. Tinbergen [2]

3. DYNAMIC MODELS AND GROWTH THEORY

Economic growth models are dynamic models. If we call a model a dynamic model, we are mainly interested in the formal structure of the model, and if we call a model an economic growth model, we are mainly interested in the economic relations between the variables of the model. A simple growth model of the post-Keynesian-type may demonstrate this distinction. One of the results of the Harrod growth model[1] is the equation:

$$y(t) = y_0 e^{s\beta t}$$

where:

y = net income
s = saving income ratio
β = accelerator

which follows from the fact that we describe the considered economy as the dynamic system:

$$\frac{1}{\beta}\dot{y} = sy.$$

This dynamic model is nothing but a logical consequence of the following assumptions:

1. Harrod

Saving is a linear function of net income, investments are linear functions of the growth rate of the net income, and saving always equals investment. The essentials of Harrod's growth theory are the economic arguments supporting these assumptions.

A dynamic model, deduced in such a way, can be the base for a statistical test of the underlying growth theory.

If we are only interested in the description of an economic growth process, we can sometimes restrict ourselves to simple dynamic models like:

$$\dot{y} = \alpha y, \qquad \alpha > 0$$

and try to estimate the coefficient a. If we attained a satisfactory agreement between the estimated and the empirical values, the dynamic model would be sufficient for the purpose of description. Detailed economic arguments for the validity of these dynamic models are still to be found, for example in Harrod's growth theory.

We once again consider a simple dynamic model, assuming the technological relation:

$$\beta k = y \qquad \beta = \text{const.}$$

where k denotes the capital stock.

We further assume that saving equals investment and again get the expression:

$$y(t) = y_0 \, e^{\beta s t}$$

Now, however, the productivity of capital b is a given constant. The growth of the economy, which is described by this dynamic model, is determined by the saving income ratio s. If it is intended to reach a fixed net income y_T at time T, then we need a special saving income ratio $\hat{s}$ given by:

$$\hat{s} = \frac{\ln(y_T/y_0)}{\beta T}.$$

If $\hat{s}$ is to be an instrumental variable of economic policy,[1] then we have to take into consideration certain economic constraints, as for example:

$$0 \leqq \hat{s} \leqq 1.$$

1. Tinbergen [3], pp. 6-8.

It is easy to see that this condition is not fulfilled if the y_T chosen is too high. In such case, it is impossible to realize this plan. The purpose of such a dynamic model is to determine this saving income ratio $\hat{s}$ for given data, which enables the economy to reach the planned net income y_T. The realisation of the saving income ratio $\hat{s}$, thus determined, is a problem of economic policy. In the theory of economic policy practical possibilities that influence the saving income ratio, are discussed,[1] such as for example: compulsory saving, tax increase, etc.

In spite of their simplicity, from a political and theoretical point of view, these models way be tools for the analysis of economic plans and past periods of economic development.

1. Tinbergen [4]

5. CONTROLLABLE SYSTEMS

In a dynamic model such as:

$$\dot{y} = \beta s y$$

with a constant β, the only parameter is s. The dynamic process evolving according to this differential aquation depends on the chosen s. In this case, the process is controlled by the magnitudes s and, therefore, we call such a dynamic system a controllable system. To generalise this idea, we assume the saving income ratio to be dependent on time:

$$0 \leq s(t) \leq 1, \qquad t \in [0,T].$$

We then get the differential equation:

$$\dot{y}(t) = \beta s(t) y(t)$$

with the solution for $y(0) = y_0$.

$$y(t) = y_0 \, e^{\beta \int_0^t s(\tau) d\tau}$$

The function s(t) is called the control function of the controllable system. In this case, there exists an infinity of different control functions which realize the aim:

$$y(T) = y_T.$$

6. OPTIMAL CONTROLS

If different ways exist for realising the aims of an economic plan, we need a criterion for the selection of a particular way. For the model just considered, a possible criterion is the maximization of the consumption in the period (0, T). That means, we have to determine a function $\hat{s}(t)$, which realises the planned y_T and maximizes

$$\int_0^T [1 - \hat{s}(t)]y(\hat{s},t)\, dt.$$

We call this integral the objective function. A further interesting objective function is the length of the planning period. In this case we have to determine the control function in such a way that we realize the aims of the plan in minimum time. The optimality so defined will express the economic efficiency of the plan.

7. MODELS FOR ECONOMIC DEVELOPMENT

The dynamic models considered deal with economies whose bottlenecks are capital assets. We assume that there are no bottlenecks in the skilled and unskilled manpower. These assumptions will only be valid for some countries and for certain time periods. But they are valid for some developing countries, and so we can use them for the description of their economic development. Then the only relevant variables are the capital stocks, and the growth path of the capital stocks describes the dynamic system completely.

We can dinstinguish models considering one or more sectors of production and models considering the reception or nonreception of economic aid.

However, in the following, we only discuss models with one resp. two sectors of production without economic aid.

8. SOME REMARKS CONCERNING MATHEMATICAL ECONOMIC MODELS

From the statistics we are acquainted with aggregative measures for the capital assets in the economy. If we want to describe the temporal development of the capital assets by a mathematical model, the economist is not primarily interested in the mathematical character of the functions in the model. On the other hand, the mathematical formulation of the model is relevant insofar as it already determines *a priori* the set of the solution functions. To put it in another way, to each mathematical specification of an economic model certain types of solution functions correspond. Some problems of the neoclassical growth models have their origin in this very fact.

In the following we consider continuous dynamic models. In this way, we can use a powerful mathematical theory for the solution of problems of optimal economic planning, namely the maximum principle of Pontryagin.[1] We describe the temporal development of the economy by continuous differentiable functions and use piece-wise continuous functions for the control functions. These restrictions seem to be reasonable for these kind of models.

1. Pontryagin

9. MATHEMATICAL DESCRIPTION OF OPTIMAL DYNAMIC PROCESSES

The following form of the outline of the maximum principle is adapted to the use of the models of economic development discussed here. For a more general outline, the reader is referred to the specific mathematical monographs.[1]

Let

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

be the vector of the variables of the dynamic system described by a system of differential equations of the first order

$$(1) \quad \begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), \dots, x_n(t); u_1(t), \dots, u_m(t)) \\ &\cdots\cdots\cdots\cdots \\ \dot{x}_n(t) &= f_n(x_1(t), \dots, x_n(t); u_1(t), \dots, u_m(t)) \end{aligned}$$

where

$$u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{pmatrix}$$

---

1. Pontryagin, Fan, Leitman

is the vector of the control functions of the system.

Introducing the vector

$$f(x,u) = \begin{pmatrix} f_1(x_1(t), \ldots, x_n(t); u_1(t), \ldots, u_m(t)) \\ f_n(x_1(t), \ldots, x_n(t); u_1(t), \ldots, u_m(t)) \end{pmatrix}$$

we can write for the system of differential equations

$$\dot{x}(t) = f(x(t), u(t)).$$

Clearly, each solution x(t) of this system depends on the control function u(t). Therefore we best denote the solution by

$$x(u,t).$$

The solutions are required to satisfy the initial condition

$$\begin{pmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{pmatrix} = \begin{pmatrix} x_{10} \\ \vdots \\ x_{n0} \end{pmatrix} = x_0$$

and the final condition

$$\begin{pmatrix} x_1(T) \\ \vdots \\ x_n(T) \end{pmatrix} = \begin{pmatrix} x_{1T} \\ \vdots \\ x_{nT} \end{pmatrix} = x_T .$$

Economically, x_{i0} means for example the existing capital stock in sector (i) at the beginning of the period, while x_{iT} means the planned capital stock in the same sector at the end of the period.

The functions

$$f(x,u)$$

and

$$f_0(x,u)$$

are assumed to be continuous in the variables x_1, $x_2, .. x_0$ and continuously differentiable with respect to $x_1, x_2, ..., x_n$.

The control functions are piecewise continuous, that means, they are continuous for all t under consideration, with the exception of only a finite number of t. For all $t \in [0,T]$, the set $U = \{u(t)\}$ is a closed subset of R^m, independent of t. The set of control functions satisfying these conditions is called the set of admissible control functions, and the problem is to find out an optimal one.

The objective function $x_0(u,t)$ is defined by

$$x_0(u,t) = \int_0^t f_0(x(\tau), u(\tau))\, d\tau.$$

An optimal control of a dynamic system

is a control function $\hat{u}$, so that the solution

$$x(\hat{u}, t)$$

satisfies the initial and final conditions

$$x(\hat{u}, 0) = x_0$$

$$x(\hat{u}, T) = x_T$$

and minimizes the objective function, that means

$$x_0(\hat{u}, T) = \min_{u \in U} x_0(u, T).$$

A different question is the so-called time minimum problem. In this special case we have

$$f_0(x(t), u(t)) \equiv 1$$

and the objective function is[1]

$$x_0 = \int_0^t f_0 x(t), \ u(t)) = \int_0^t d\tau.$$

That means, the control function minimizes the transition time from the initial state x(0) to the final state x(T).

1. Pontryagin, pp. 26

10. THE MAXIMUM PRINCIPLE

Let us now proceed to the formulation of the maximum principle, which gives us the solution of the problems discussed. In addition to the system of differential equations(1) we introduce a system of auxiliary functions

$$\psi(t)=\begin{pmatrix}\psi_0(t)\\ \psi_1(t)\\ \vdots\\ \psi_n(t)\end{pmatrix}$$

which are solutions of the system of differential equations of the first order

$$-\dot{\psi}_1=\sum_{k-0}^{n}\frac{\partial f_k(x,u)}{\partial x_i}\psi_k, \qquad i=0,1,\ldots,n.$$

Now we combine the two systems of functions ψ_i and x_i into one expression which we call the Hamilton function of the system and which is defined by

$$(2)\quad H(\psi,x,u)=\sum_{k=0}^{n}\psi_k f_k(x,u).$$

In this notation, we can write for the differential equation

$$(3)\quad \dot{\psi}_i=-\frac{\partial H}{\partial x_i}, \qquad i=0,1,\ldots,n.$$

In general, the auxiliary functions have to satisfy certain conditions. For instance, in the case where the integral

$$(4) \qquad x_0(u,t) = \int_0^T f_0(x(t), u(t))\, dt$$

is to be minimized with fixed initial and final conditions x(0) and x(T), these are

$$\psi_0(T) \leq 0$$

and

$$\psi(t) \neq (0, 0, \ldots, 0).$$ [1]

Now we consider the Hamilton function as a function of the m-tupel

$$u \in U$$

only, ψ and x being fixed values. Let us denote the least upper bound of the values of these functions by M (ψ, x)

$$(5) \qquad M(\psi,x) = \sup_{u \in U} H(\psi,x,u)$$

$$= H(\psi,x,u).$$

The maximum principle says: Under certain conditions, such as the preceding ones, where we minimized the integral (4) under

1. Pontryagin

the condition of fixed initial and final state of the variables x(t) in a closed time interval, the necessary condition for a control function u to be optimal is that u is a component of the solution vector

$$(\psi, x, H, u)$$

of the equation systems (1), (2), (3), (4) and (5). Therefore, the maximum principle only gives a necessary condition. Only for the special case where we have to determine the optimal control of a linear time-optimal process, there exists proof of the existence and uniqueness of the solution. In the case that all relevant functions are concave with respect to the controls, the existence of optimal paths can be proved. To a certain degree, our mathematical considerations are unsatisfactory. The existence and uniqueness of an optimal control function cannot be proved. We assume, that if the solution found is economically reasonable, this solution is indeed the optimal one.

11. SOME MODELS OF OPTIMAL CAPITAL ACCUMULATION

In the following we consider some models for an optimal capital accumulation. We do not consider the problem of allocation of labour input. The models are formulated in per capita quantities. This formulation makes the results economically more plausible; besides, it is easier to be handled mathematically.

We regard models with linear production functions (model I, III, IV) and a model with a neoclassical production function (Model II). We regard models with one sector of production (model I and II) and such with two sectors of production (model III and IV). In some models (model I, II and III) we discuss the problem of maximization of consumption in a period and in one model the minimum-time problem. We develop the problems in the following sequence:

- —Statement of the general economic problem
- —Formulation of the corresponding model and the objective function.
- —Procedure according to Pontryagin
- —Solution and numerical example.

The readers interested in the mathematics involved will find some hints in the Appendix. However, we dispense with detailed proofs.

The symbols used in the following models are:

Υ net national product
C consumption
K investment
N size of population
y per capita net national product
k per capita capital stock
c per capita consumption
γ productivity of capital
α elasticity of production of the capital
π rate of population increase

12. MODEL I: A ONE-SECTORAL MODEL WITH LINEAR PRODUCTION FUNCTION

We assume that the net product is a linear function of the capital stock. We further assume the capacity of the capital stock to be fully used all the time and that saving always equals investment. The economy produces only one kind of goods, which can be used either for consumption or investment. The population increases by a constant rate. A certain fixed minimal per capita consumption shall be realized within the whole planning period.

For this reason, the available net product is supposed to be positive at the beginning of the planning period. If this assumption does not hold, it is impossible to speak of a process of capital accumulation at all.

Let the macro-economic production function be

$$Y = \gamma K.$$

Together with the assumption that saving equals investment

$$S = \dot{K}$$

the equation of income-spending

$$Y = C + \dot{K}$$

follows. After division by the size N of population, the per capita production function is

$$y = \gamma k.$$

In the same way follows

$$y = \frac{C}{N} + \frac{\dot{K}}{N}$$

$$= \frac{C}{N} + \frac{\dot{K}N}{N^2} - \frac{K\dot{N}}{N^2} + \frac{K}{N}\frac{\dot{N}}{N}$$

$$= c + \dot{k} + k\frac{\dot{N}}{N}.$$

If the rate of population increase $\frac{\dot{N}}{N} = \pi$ is a constant, as we assume, we have

$$y = c + \dot{k} + \pi k \cdot$$

Then the per capita consumption is

$$c = (\gamma - \pi) k - \dot{k}.$$

$$\gamma > \pi$$

Taking into regard the fixed minimum per-capita consumption $\bar{c}$, the available net pro duct is

$$y - \pi k - \bar{c} = (\gamma - \pi) k - \bar{c}.$$

Let u(t) be the proportion of the available product which we use for capital accumulation and (1 - u(t)) the proportion of the available product which is consumed, then the allocation of the available product can be described by the system

$$\dot{k} = u[(\gamma - \pi) k - \bar{c}], \quad 0 \leq u(t) \leq 1$$

(6) $c^* = c - \bar{c} = (1 - u)[(\gamma - \pi) k - \bar{c}]$.

Let the initial per capita capital stock be

$$k_0 = k(0)$$

and the planned final per capita capital stock at time point T be

$$k_T = k(T).$$

The objective $k_T = k(T)$ is to be realized in such a manner that the total per capita consumption is maximized during the planning period (0,T), that is

$$\int_0^T c\,dt = \int_0^T (c^* + \bar{c})\,dt \overset{!}{=} \max.$$

Obviously, it is sufficient to maximize the integral

$$\int_0^T c^* \, dt$$

or, which is the same thing to minimize the integral

$$\int_0^T - c^* \, dt = \int_0^T (u - 1)[\gamma - \pi) k - \bar{c}] \, dt.$$

The proportion u(t) is the control function of the process, described by the system (6).

According to Pontryagin,[1] the optimal control function u(t) is a component of the solution vector $(x, k, \psi_0, \psi_1, H, u)'$ of the equation system

$$\dot{x} = (u - 1)[(\gamma - \pi) k - \bar{c}]$$

$$\dot{k} = u[(\gamma - \pi) k - \bar{c}]$$

(7) $$-\dot{\psi}_0 = \frac{\partial H}{\partial x}$$

(8) $$-\dot{\psi}_1 = \frac{\partial H}{\partial k}$$

(9) $$H = (u - 1)[(\gamma - \pi) k - \bar{c}]\psi_0 + u[(\gamma - \pi) k - \bar{c}]\psi_1$$

1. Pontryagin L.S., pp. 66

$$(10) \qquad H(\bar{u}) = \sup_{\bar{u} \in U} H(u)$$

under the conditions

$$(\psi_0(t), \psi_1(t)') \not\equiv (0,0)'$$

and

$$(11) \qquad \psi_0(T) \leqq 0.$$

From (7) and (8) follows together with (9)

$$(12) \qquad \dot{\psi}_0 \equiv 0$$

and

$$\dot{\psi}_1 = -(u-1)(\gamma-\pi)\psi_0 - u(\gamma-\pi)\psi_1.$$

From (12) follows

$$(13) \qquad \psi_0 \equiv \text{const.}$$

and regarding (11) we can write

$$(14) \qquad \psi_0 \equiv -1.$$

Together with (9) and (14), (10) yields

$$(1-u)[\gamma-\pi)k-\bar{c}] + u[(\gamma-\pi)k-\bar{c}]\psi_1$$

$$= \sup_{\bar{u} \in U} \{(1-\bar{u})[(\gamma-\pi)k-\bar{c}] + \bar{u}[(\gamma-\pi)k-\bar{c}]\psi_1\}$$

$$= [(\gamma-\pi)k-\bar{c}] + \sup_{\bar{u} \in U} \{\bar{u}[(\gamma-\pi)k-\bar{c}](\psi_1-1)\}.$$

The optimal control function thus takes on the values zero or one only, depending on whether the sign of $[(\gamma - \pi)k - \overline{c}]\,(1 - \psi_1)$ is negative or positive.

The optimal control function is

$$u(t) = \begin{cases} 1 \text{ for } t \in (0, t^*) \\ 0 \text{ '' } \; t \in (t^*, T) \end{cases}.$$

Then the optimal per capita path is

$$(15) \quad k(t) = \begin{cases} \left(k_0 - \dfrac{\overline{c}}{\gamma - \pi}\right) e^{(\gamma - \pi)t} + \dfrac{\overline{c}}{\gamma - \pi} & \text{for } t \in (0, t^*) \\ K_T & \text{for } t \in (t^*, T) \end{cases}.$$

For the optimal per capita consumption path we get

$$c(t) = \begin{cases} \overline{c} & \text{for } t \in (0, t^*) \\ (\gamma - \pi)\, k_T & \text{for } t \in (t^*, T) \end{cases}.$$

Observing the boundary conditions K_0 and k(T) we can derive the time point t* from(15).

$$t^* = \frac{1}{\gamma - \pi} \ln \frac{(\gamma - \pi)\, k_T - \overline{c}}{(\gamma - \pi)\, k_0 - \overline{c}}.$$

It is possible for a t* thus calculated to lie outside of the planning period

$$t^* \notin [0, T].$$

In this case, it is impossible to realize the planned final per capita capital stock. So the existence of a solution t^* with $t^* \in (0,T)$ is a criterion for the solvability of the economic problem considered.

For example, with the data

$$k_0 = 14$$
$$k_T = 25$$
$$\bar{c} = 4$$
$$\gamma = 0.35$$
$$\pi = 0.025$$

and $T=20$ the following paths can be found:

t	k(t)	c(t)
0	14,000	4,000
1	14,650	
2	15,549	
3	16,794	
4	18,517	
5	20,901	
6	24,200	
t*	25,000	4,000
7		8,125
8		
9		
10		
11		
12		
13		
14		
15		
16		
17		
18		
19		
20	25,000	8,125

13. MODEL II : A ONE-SECTORAL MODEL WITH A NEOCLASSICAL PRODUCTION FUNCTION

The aim again is to transform the economy within a given time period (0,T) from an initial state described by the initial capital stock k_0 into a final state, which is described by a given final capital stock k_T, in such a way that the per capita consumption is maximized during the planning period. We assume that the law of production is governed by a neoclassical production function, that is, a production function with positive, but diminishing marginal products,

Let α be the production elasticity of the capital, then it is possible, to write a Cobb-Douglas function in per capita magnitudes

$$y = Fk^{\alpha},$$

if the factor labour is proportional to the population. F is a constant. If the population increases by a constant rate, the per capita consumption is

$$c = Fk^{\alpha} - \pi k - \dot{k}.$$

The objective function is

$$(16) \qquad \int_0^T c \, dt.$$

A minimum consumption which shall be proportional to the production

$$\bar{c} = \lambda y = \lambda F k^{\alpha}, \qquad \text{with } 0 < \lambda < 1$$

is always to be guaranteed. As available product therefore remains

$$y - \lambda y - \pi k = [(1 - \lambda) F k^{\alpha} - \pi k].$$

The available product can be used for consumption to increase consumption or can be used to increase the per capita capital stock. If the additional consumption is symbolized again by c*, it is

$$c^* + \dot{k} = [1 - \lambda) F k^{\alpha} - \pi k].$$

A piecewise continuous function u(t)

$$u(t) \in U := \{u(t) \mid 0 \leq u(t) \leq 1\}$$

controls the distribution of the available product into c* and k*

$$c^* = \quad u\,[(1 - \lambda) F k^{\alpha} - \pi k]$$

$$\dot{k} = (1 - u\,[(1 - \lambda) F k^{\alpha} - \pi k].$$

The control function u(t) is now to be determined in such a way, that the k-path satisfies the boundary values k_0 and k_T and the integral (16) takes a maximum. The maximization of this integral equals the minimization of the integral

$$\int_0^T \left(-u\left[(1-\lambda)Fk^{\alpha} - \pi k\right] - \lambda F k^{\alpha} \right) dt.$$

According to Pontryagin[1] the desired optimal control function u(t) is a component of the solution vector $(k, x, H, \psi_0, \psi_1, u)'$ of the equation system

$$\dot{k} = (1-u)\left[(1-\lambda)Fk^{\alpha} - \pi k\right]$$

$$\dot{x} = \quad -u\left[(1-\lambda)Fk^{\alpha} - \pi k\right] - \lambda F k^{\alpha}.$$

(17) $$H = \psi_0 \dot{x} + \psi_1 \dot{k}$$

(18) $$\dot{\psi}_0 = -\frac{\partial H}{\partial x}$$

$$\dot{\psi}_1 = -\frac{\partial H}{\partial k}$$

(19) $$H(u) = \sup_{\bar{u} \in U} H(\bar{u}).$$

The component of the solution u(t) is the wanted optimal control under the conditions

$$(\psi_0, \psi_1)' \neq (0, 0)'$$

1. Pontryagin, p. 66

and

$$\psi_0(T) \leqq 0.$$

The functions ψ_0 and ψ_1 are determined only up to a multiplicative constant. Because of (17) and (18) it is

$$\dot{\psi}_0 \equiv 0,$$

and therefore it can be written

$$\psi_0 \equiv -1. \tag{20}$$

For (19) together with (20) it is

$$u(1-\psi_1)[(1-\lambda)Fk^\alpha - \pi k] = \sup_{\bar{u} \in U} \{\bar{u}(1-\psi_1)[(1-\lambda)Fk^\alpha - \pi k]\}.$$

It follows that the optimal control function takes on the values zero or one only, depending on whether the sign of $(1-\psi_1)$ and $[(1-\lambda)Fk^\alpha - \pi k]$ is the same or not. The desired optimal k=path consists of pieces with

$$k \equiv \text{const.} \tag{21}$$

and solutions of the differential equation

$$\dot{k} = (1-\lambda)Fk^\alpha - \pi k. \tag{22}$$

If the optimal control function is known, the k=path is determined

In this case the only control, which is compatible with the equation system and therefore is the optimal control, takes the value zero from $t=0$ until a point $t=t^*$, and from that point jumps to the value one.

The optimal growth path follows from (21) and (22).

$$k(t)=\begin{cases}\left\{\left[k_0^{1-\alpha}-\frac{F}{\pi}(1-\lambda)\right]e^{-\pi(1-\alpha)t}+\frac{F}{\pi}(1-\lambda)\right\}^{\frac{1}{1-\alpha}}, & \text{for } 0<t<t^* \\ k_T, & \text{for } t^*<t<T.\end{cases}$$

During the first phase, consumption consists only of minimum consumption, during the second phase of the product reduced by the costs of the increasing population. So, for the path of the optimal per capita consumption it follows:

$$c=\begin{cases}\lambda F\left\{\left[k_0^{1-\alpha}-\frac{F}{\pi}(1-\lambda)\right]e^{-\pi(1-\alpha)t}+\frac{F}{\pi}(1-\lambda)\right\}^{\frac{\alpha}{1-\alpha}}, & \text{for } 0<t<t^* \\ Fk_T^{\alpha}-\pi k_T & \text{for } t^*<t<T.\end{cases}$$

Taking regard of the boundary values k_0 and k_T it follows

$$t^*=(-\pi(1-\alpha))^{-1}\ln\frac{k_T^{1-\alpha}-\frac{F}{\pi}(1-\lambda)}{k_0^{1-\alpha}-\frac{F}{\pi}(1-\lambda)}.$$

With the values

$$k_0 = 14$$
$$k_T = 25$$
$$\lambda = 0.70$$
$$\alpha = 0.75$$
$$F = 0.70$$
$$\pi = 0.025$$

and $T = 20$ the following paths for a development maximizing the consumption can be found:

t	k(t)	d(t)
0	14,000	3,546
1	15,203	3,773
2	16,474	4,007
3	17,814	4,249
4	19,226	4,499
5	20,710	4,757
5	22,268	5,023
7	23,902	5,297
t*	25,000	5,478
8	25,000	7,201
9		
10		
11		
12		
13		
14		
15		
16		
17		
18		
19		
20	25,000	7,201

In the following table the consequences of isolated changes of single magnitudes on the time point t* are demonstrated:

k_0	k_T	λ	α	F	π	t*
14	25	0.70	0.75	0.70	0.020	7,127
					0.025	7,647
					0.030	8,188
			0.80	0.60	0.025	7,720
		0.80	0.75	0.70		13,745
	23	0.70				6,454
	27					8,779

14. MODEL III: MAXIMIZATION OF CONSUMPTION IN A TWO-SECTORAL MODEL

A two-sector model is now regarded. Sector 1 only produces capital goods and Sector 2 only produces consumption goods. At the beginning of the planning period certain fixed capital stocks exist in both sectors, and for the end of the planning period certain fixed capital stocks are planned. In the models discussed until now the problem was always to decide between consumption and investment. Now it is to decide between investment in the sector of capital goods and the sector of consumption goods. The net product Y_1 of sector 1 is used for replacement of capital assets μY_2 in sector 2 and for the rates of change $\dot{K}_1$, $\dot{K}_2$ in the stocks of capital in both sectors

$$Y_1 = \mu Y_2 + \dot{K}_1 + \dot{K}_2$$
$$Y_2 = C.$$

In both sectors the processes of production are described by

$$Y_1 = \gamma_1 K_1$$
$$Y_2 = \gamma_2 K_2.$$

The accumulation of capital is to be controlled by the current investment decisions in such a manner as to maximize the sum of the per capita consumption in the period from $t=0$ to $t=T$,

$$\int_0^T \frac{C}{N} dt = \int_0^T c\, dt,$$

and leading the initial sectoral capital stocks up to the wanted final capital stocks. The capital invested in any one of the sectors is not supposed to be transferred into the other sector. Following this, it is discussed in model IV, what kind of control of the investment decisions leads the initial capital stocks to the desired final capital stocks within a minimum time period. The solutions of the minimum time problem and the maximization of consumption are confronted in a numerical illustration.

With a constant rate of increase π of the population size N, together with

$$\frac{\dot{K}_i}{N} = \dot{k}_i + \pi k_i, \qquad i = 1, 2,$$

$$\frac{Y_i}{N} = y_i, \qquad i = 1, 2,$$

the model in per capita quantities results in

$$y_1 = \mu y_2 + \dot{k}_1 + \dot{k}_2 + \pi(k_1 + k_2)$$
$$y_2 = c$$
$$y_1 = \gamma_1 k_1$$
$$y_2 = \gamma_2 k_2.$$

A piecewise continuous function $u(t)$

$$u(t) \in U := \{u(t) \mid 0 \leqq u(t) \leqq 1\}$$

in this context controls the rate of change of the per capita capital stocks in both sectors.

$$\dot{k}_1 = \quad u\,[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2]$$
$$\dot{k}_2 = (1 - u)\,[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2]$$

maximizing

$$\int_0^T c\,dt = \int_0^T \gamma_2 k_2\,dt = \gamma_2 \int_0^T k_2\,dt$$

resp. minimizing

$$\gamma_2 \int_0^T -k_2\,dt.$$

The paths of development $k_1(t)$ and $k_2(t)$ have to satisfy the boundary conditions $k_1(0) = k_{10}$, $k_2(0) = k_{20}$, $k_1(T) = k_{1T}$ and $k_2(T) = k_{2T}$.

In this model the available product, which is the net product of sector 1 minus the replacement of capital assets and the costs of population growth, is only taken to increase the per capita capital stocks. If the initial available product is positive, then we get - as is shown later - per capita capital stocks which increase monotonously in both sectors.

According to Pontryagin the optimal control function u(t) is a component of the solution vector $(x, k_1, k_2, \psi_0, \psi_1, \psi_2, H, u)'$ of the equation system

$$\dot{x} = -k_2$$

(23) $$\dot{k}_1 = u[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2]$$

(24) $$\dot{k}_2 = (1 - u)[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2]$$

(25) $$-\dot{\psi}_0 = \frac{\partial H}{\partial x}$$

$$-\dot{\psi}_1 = \frac{\partial H}{\partial k_1}$$

$$-\dot{\psi}_2 = \frac{\partial H}{\partial k_2}$$

(26) $$H = -k_2\psi_0 + u[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 - \pi)k_2]\psi_1 + (1 - u)[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2]\psi_2$$

(27) $$H(u) = \sup_{\bar{u} \in U} H(\bar{u}).$$

Further necessary conditions so that the control function u(t) solves this optimum problem are:

$$(\psi_0(t), \psi_1(t), \psi_2(t))' \neq 0,0,0)'$$

and

$$\psi_0(T) \leqq 0.$$

As the function H is homogeneous in ψ_0, ψ_1 and ψ_2, these functions are only determined up to a multiplicative constant.

From (25) follows

$$\psi_0 \equiv \text{const.}$$

Therefore regarding (28) we can write

$$(29) \qquad \psi_0 \equiv -1.$$

Together with (26) and (29), (27) yields

$$u(\psi_1 - \psi_2)[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2]$$
$$= \sup_{\bar{u} \in U} \{\bar{u}(\psi_1 - \psi_2)[\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2]\}.$$

The optimal control function u(t) thus takes on the values zero or one only, depending on whether the sign of $(\psi_1 - \psi_2)[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 - \pi)k_2]$ is negative or positive.

So the desired control function of the accumulation of capital maximizing consumption is

$$(30)\qquad u(t)=\begin{cases} 1 \text{ for } t \in (0,t^{**}) \\ 0 \text{ for } t \in (t^{**},t^{*}) \\ 1 \text{ for } t \in (t^{*},T). \end{cases}$$

Therefore from (23) follows for $t \in (0,t^{**})$

$$(31)\qquad \dot{k}_1=(\gamma_1-\pi)k_1-(\mu\gamma_2+\pi)k_2$$

and from (24)

$$\dot{k}_2 \equiv 0.$$

Then it is

$$k_2 \equiv k_2(0)=k_2(t^{**})=k_{20},$$

and the solution of the differential equation (31) is

$$(32)\qquad k_1=\left(k_{10}-\frac{(\mu\gamma_2+\pi)}{(\gamma_1-\pi)}k_{20}\right)e^{(\gamma_1-\pi)t} + \frac{(\mu\gamma_2+\pi)}{(\gamma_1-\pi)}k_{20}.$$

Together with (30) from (24) follows for $t \in (t^{**},t^{*})$

$$(33)\qquad \dot{k}_2=(\gamma_1-\pi)k_1-(\mu\gamma_2+\pi)k_2$$

and from (23)

$$\dot{k}_1 \equiv 0.$$

Then it is

$$k_1 \equiv k_1(t^{**}) = k_1(t^*) = k_{1*},$$

and the differential equation (33) has the solution

$$(34) \qquad k_2 = (k_{20} - \frac{(\gamma_1 - \pi)}{(\mu\gamma_2 + \pi)} k_{1*}) e^{-(\mu\gamma_2 + \pi(t - t^{**}))} + \frac{(\gamma_1 - \pi)}{(\mu\gamma_2 + \pi)} k_{1*}.$$

From (30) follows together with (23) for $t \in (t^*, T)$

$$(35) \qquad \dot{k}_1 = (\gamma_1 - \pi) k_1 - (\mu\gamma_2 + \pi) k_2$$

and from (24)

$$\dot{k}_2 \equiv 0.$$

Then it is

$$k_2 \equiv k_2(T) = k_2(t^*) = k_{2T},$$

and the differential equation (35) has the solution

$$k_1 = k_{1T} - \frac{(\mu\gamma_2 + \pi)}{(\gamma_1 - \pi)} k_{2T}) e^{(\gamma_1 - \pi)(t - T)} + \frac{(\mu\gamma_2 + \pi)}{(\gamma_1 - \pi)} k_{2T}.$$

The points of time t* and t** can be determined by the relations (32), (34) and (36). We will find for t**

$$t^{**} = \frac{1}{(\gamma_1 - \pi)} \left\{ \ln \frac{k_{2T} - k_{20}}{\frac{k_{10}(\gamma_1 - \pi)}{(\mu\gamma_2 + \pi)} - k_{20}} \right.$$

$$\left. - \ln \left(1 - \left(\frac{k_{1T} - \frac{(\mu\gamma_2 + \pi)}{(\gamma_1 - \pi)} k_{2T}}{k_{10} - \frac{(\mu\gamma_2 + \pi)}{(\gamma_1 - \pi)} k_{20}} \right)^{\frac{(\mu\gamma_2 + \pi)}{(\gamma_1 + \mu\gamma_2)}} e^{-\frac{(\gamma_1 - \pi)(\mu\gamma_2 + \pi)T}{(\gamma_1 + \mu\gamma_2)}} \right) \right\}$$

and for t*

$$t^* = t^{**} + \frac{(\gamma_1 - \pi)}{(\gamma_1 + \mu\gamma_2)} T - \frac{1}{(\gamma_1 + \mu\gamma_2)} \ln \frac{(\gamma_1 - \pi)k_{1T} - (\mu\gamma_2 + \pi)k_{2T}}{(\gamma_1 - \pi)k_{10} - (\mu\gamma_2 + \pi)k_{20}}$$

So the desired paths of development are completely determined. For $k_1(t)$ follows

$$k_1(t) = \left(k_{10} - \frac{(\mu\gamma_2 + \pi)}{(\gamma_1 - \pi)} k_{20} \right) e^{(\gamma_1 - \pi)t} + \frac{(\mu\gamma_2 + \pi)}{(\gamma_1 - \pi)} k_{20} \qquad \text{for } t \in (0, t^{**})$$

$$k_1(t) = \left(k_{10} - \frac{(\mu\gamma_2 + \pi)}{(\gamma_1 - \pi)} k_{20}\right) e^{(\gamma_1 - \pi)t^{**}}$$

$$+ \frac{(\mu\gamma_2 + \pi)}{(\gamma_1 - \pi)} k_{20} = k_{1*} \qquad \text{for } t \in (t^{**}, t^*)$$

$$k_1(t) = \left(k_{1T} - \frac{(\mu\gamma_2 + \pi)}{(\gamma_1 - \pi)} k_{2T}\right) e^{(\gamma_1 - \pi)(t - T)}$$

$$+ \frac{(\mu\gamma_2 + \pi)}{(\gamma_1 - \pi)} k_{2T} \qquad \text{for } t \in (t^*, T).$$

For $k_2(t)$ follows

$$k_2(t) = k_{20} \qquad \text{for } t \in (0, t^{**})$$

$$k_2(t) = \left(k_{20} - \frac{(\gamma_1 - \pi)}{(\mu\gamma_2 + \pi)} k_{1*}\right) e^{-(\mu\gamma_2 + \pi)(t - t^{**})}$$

$$+ \frac{(\gamma_1 - \pi)}{(\mu\gamma_2 + \pi)} k_{1*} \qquad \text{for } t \in (t^{**}, t^*)$$

$$k_2(t) = k_{2T} \qquad \text{for } t \in (t^*, T).$$

The accumulation of capital, maximizing the sum of per capita consumption, is characterized by the fact that first, the per capita capital stock is increased in the sector of capital goods exclusive'y. Second, only the per capita stock in the sector of consumption goods is increased until a given final value is reached. Third and last, only the per capita capital stock in the sector of capital goods is increased until a given final value is reached.

In the following. the influence of the length of the different phases of accumulation caused by different objectives shall be demonstrated by a numerical example. The data assumed here follow in its dimensions the Algerian model by Stoleru.[1]

With

$$k_{10} = 4 \qquad k_{1T} = 10$$

$$k_{20} = 10 \qquad k_{2T} = 15$$

$$\gamma_1 = 0.25 \qquad \gamma_2 = 0.4$$

$$\mu = 0.1 \qquad \pi = 0.025$$

and T=20 in the problem of maximizing consumption, the following paths of development are found:

1. Stoleru

t	$k_1(t)$	$k_2(t)$	$c(t)$
0	4.00	10.00	4.00
1	4.28		
2	4.63		
3	5.07		
4	5.62		
t^{**}	5.93	10.00	4.00
5		10.35	4.14
6		10.99	4.40
7		11.59	4.64
8		12.16	4.86
t_1		12.29	4.92
9		12.68	5.07
10		13.18	5.27
11		13.64	5.46
t_0		13.97	5.59
12		14.07	5.63
13		14.48	5.79
14		14.86	5.94
t^*	5.93	15.00	6.00
15	6.17		
16	6.64		
17	7.22		
18	7.95		
19	8.86		
20	10.00	15.00	6.00

In the following table, the consequences of isolated changes of single magnitudes on the time points t* and t** are demonstrated

k_{10}	k_{1T}	k_{20}	k_{2T}	γ_1	γ_2	μ	π	T	t**	t*
4	10	10	15	0.25	0.5	0.10	0.025	20	4.48	14.38
5	—	—	—	—	—	—	—	—	1.01	13.12
4	12	—	—	—	—	—	—	—	4.84	13.70
—	10	12	—	—	—	—	—	—	6.45	13.82
—	—	10	17	—	—	—	—	—	5.86	16.13
—	—	—	15	0.30	—	—	—	—	1.02	13.16
—	—	—	—	0.25	0.4	—	—	—	4.48	14.38
—	—	—	—	—	0.5	0.12	—	—	14.27	19.21
—	—	—	—	—	—	0.10	0.030	—	11.56	17.80
—	—	—	—	—	—	—	0.025	15	9.60	14.13

15. MODEL IV : THE MINIMUM TIME PROBLEM IN A TWO-SECTOR MODEL

Now for this model the control function u(t) is to be determined which leads the initial sectoral capital stocks into the wanted final capital stocks within a minimum time period $(0,t_0)$. According to Pontryagin[1] this optimal control function u(t) is a component of the solution vector $(k_1, k_2, \psi_1, \psi_2, H, u)'$ of the equation system

$$\dot{k}_1 = u\,[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2],$$

$$\dot{k}_2 = (1 - u)\,[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2],$$

$$-\dot{\psi}_1 = \frac{\partial H}{\partial k_1},$$

$$-\dot{\psi}_2 = \frac{\partial H}{\partial k_2},$$

$$\begin{aligned} H &= u[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2]\psi_1 \\ &\quad + (1 - u)[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2]\psi_2 \\ &= u\,(\psi_1 - \psi_2)[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2] \\ &\quad + [(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2]\psi_2, \end{aligned}$$

$$H(u) = \sup_{\bar{u} \in U} H(\bar{u})$$

---

1. Pontryagin, p. 115

under the condition of

$$\psi_1(t), \psi_2(t))' \not\equiv (0,0)'.$$

Again the optimal control function takes on the value zero or one only, depending on the sign of $(\psi_1 - \psi_2)[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2]$ being negative or positive.

The wanted optimal control function is

$$u \equiv 1 \quad \text{for } t \in (0, t_1)$$
$$u \equiv 0 \quad \text{for } t \in (t_1, t_0).$$

The time points t_1 and t_0 can be calculated with the help of the given initial and final capital stocks k_{10}, k_{20}, $k_{1T} = k_{1t_1}$ and $k_{2T} = k_{2t_0}$ from

$$k_{1t_1} = \left(k_{10} - \frac{(\mu\gamma_2 + \pi)}{(\gamma_1 - \pi)} k_{20}\right) e^{(\gamma_1 - \pi)t_1} - \frac{(\mu\gamma_2 + \pi)}{(\gamma_1 - \pi)} k_{20}$$

and

$$k_{2t_0} = \left(k_{20} - \frac{(\gamma_1 - \pi)}{(\mu\gamma_2 + \pi)} k_{1t_1}\right) e^{-(\mu\gamma_2 + \pi)(t_0 - t_1)} \frac{(\gamma_1 - \pi)}{(\mu\gamma_2 + \pi)} k_{1t_1}$$

So it is found

$$t_1 = \frac{1}{(\gamma_1 - \pi)} \ln \frac{k_{1t_1} - \frac{(\mu\gamma_2 + \pi)}{(\gamma_1 - \pi)} k_{20}}{k_{10} - \frac{(\mu\gamma_2 + \pi)}{(\gamma_1 - \pi)} k_{20}}$$

and

$$t_1 - t_0 = \frac{1}{(\mu\gamma_2 + \pi)} \ln \frac{k_{2t_0} - \frac{(\gamma_1 - \pi)}{(\mu\gamma_2 + \pi)} k_{1t_1}}{k_{20} - \frac{(\gamma_1 - \pi)}{(\mu\gamma_2 + \pi)} k_{1t_1}}.$$

With the same data as in the numerical example of model III, we get the following optimal paths:

t	$k_1(t)$	$k_2(t)$	$c(t)$
0	4.00	10.00	4.00
1	4.28		
2	4.63		
3	5.07		
4	5.62		
t**	5.93		
5	6.31		
6	7.17		
7	8.26		
8	9.61		
t_1	10.00	10.00	4.00
9		11.17	4.47
10		12.65	5.06
11		14.03	5.61
t^0	10.00	15.00	6.00
12			

In the following table the effects of isolated changes of the different data concerning points t_1 and t_0 are shown.

k_{10}	k_{1T}	k_{20}	k_{2T}	γ_1	γ_2	μ	π	T	t_1	t_0
4	10	10	15	0.25	0.5	0.10	0.025	20	8.25	11.74
5	—	—	—	—	—	—	—	—	5.40	8.89
4	12	—	—	—	—	—	—	—	9.35	12.01
—	10	12	—	—	—	—	—	—	11.14	13.33
—	—	10	17	—	—	—	—	—	8.25	13.40
—	—	—	15	0.30	—	—	—	—	5.60	8.19
—	—	—	—	0.25	0.4	—	—	—	8.25	11.74
—	—	—	—	—	0.5	0.12	—	—	14.81	19.07
—	—	—	—	—	—	0.10	0.030	—	13.01	17.22
—	—	—	—	—	—	—	0.025	15	10.23	14.07

16. SOME REMARKS ON STATISTICAL PROBLEMS

If we are looking for econometric models on the basis of these models, a lot of statistical problems will arise. One of these is the evaluation of the capital stock.

In economic theory produced capital goods are simply called capital. A direct registration in quantity units is statistically an unsolvable task. It is only when referring to money units that we are able to record a capital stock for the whole economy. The development of this stock, after elimination of price changes, is of particular interest. There are different possibilities to record statistically the capital stock, for instance, estimation of the total value, registration by sampling or cumulation of investments.

Investments in plant and equipment are the flow of goods which enlarge capital assets. The real content of the term «capital assets» is defined by agreement which determines the goods belonging to it. Therefore, land is not usually taken into consideration but the cost of recovery of lands and meliorations. Furthermore, for a good to be a part of the capital stock, it must be at its final destination, and its expected useful life has to be longer than the accounting period. Produced goods satisfying these criteria, but which are purchased by private households or which serve military purposes, do not belong to capital assets.

If we derive capital assets from time series of investments - this method usually is called perpetual inventory method - we have to eliminate the effects of price-changes. By such a deflation of the investment outlays, we circumvent the direct evaluation of the stocks. In investigations like the present ones, the gross capital asset is mostly considered as a measure for the possibilities of production in the next period. According to the «perpetual inventory method», gross capital consists of the sum of the gross capital investments of former periods still available at the beginning of the period. For this method we need information about the homogeneity of the capital assets and of their physical life. It is unknown how far the capital assets thus calculated give a satisfying approximation to reality. However, it can be supposed, that the choice of a suitable survival function plays a decisive rôle. Survival functions show which part of the gross capital investment of a certain year can be used in the period considered. Empirical investigations show that linear and logistical functions can serve as a good approximation.

In the case of multisectoral models, an additional difficulty arises insofar as the statistical definitions of the sectors do not agree with the functional definition of the sectors in the models. Therefore, we need special surveys for planning. If we use published input-output tables, we have the problem of aggregating the sectors according to functional viewpoints, which requires a thorough knowledge of the statistical material.

Another problem is the estimation of the future size of a population. The simplest method is to take the number of individuals, as determined at a more or less recent date in the past, and to apply to it an assumed rate, possibly derived from observations on the past growth of the population itself, or by analogy with rates observed in other populations in similar circumstances. The really distinguishing feature of such a projection lies in the fact that calculations are applied to the figure of total population only, rather than to population segments, or relations between the population and its environment. But this kind of projection is sufficient for the models discussed here.

For application of the planning concepts developed here, the econometric quality of the structural relations is of decisive importance. One of the most delicate problems is the linearity of the production functions used. However, in favour of constant capital-output ratios we can state: There are certain arguments for a secular decline of the capital-output ratio.

However, in general, the price index of capital goods moves up more rapidly than the price index of the net product. If we deflate both magnitudes by the same index, as is usually done, we get by tendency an increase of the capital output ratio. For a planning period of about five to ten years duration, we can hope that the two effects will compensate.

17. APPENDIX : SOME FURTHER REMARKS ON THE CONSIDERED MODELS

ON MODEL I :

The proof of the optimal solution is based on the relation

$$H - [(\gamma - \pi)k - \bar{c}]$$

$$= \sup_{\bar{u} \in [0,1]} \{\bar{u}[(\gamma - \pi)k - \bar{c}](\psi_1 - 1)\}$$

from which it follows that the optimal control function takes on the values zero or one only, depending on whether the sign of $[(\gamma - \pi)k - \bar{c}]$ $(\psi_1 - 1)$ is negative or positive. If the expression $[(\gamma - \pi)k - \bar{c}]$ is positive for $t = 0$, then it follows that $[(\gamma - \pi)k - \bar{c}](\psi_1 - 1)$ is always positive. Thus the optimal control function is

$$u \equiv 1 \qquad \text{for } \psi_1 > 1$$

$$u \equiv 0 \qquad \text{for } \psi_1 < 1$$

and is undetermined for $\psi_1 - 1$. In the last case, it can be proved that there is only a finite number of such points in the planning period $(0, T)$, since, by definition, $u(t)$ is piecewise continuous in $(0,T)$. For such a point $u(t)$ can be completed by one of its limits.

The optimal control function $u(t)$ is determined successively commencing from $t=T$. There are two possibilities:

$u(t) \equiv 0$ for t in a neighbourhood $U(T)$ of $t=T$

$u(t) \equiv 1$ for t in a neighbourhood $U(T)$ of $t=T$

By discussion of these alternatives, the optimal control function $u(t)$ can be determined, taking regard of the Pontryagin system and the initial and final capital stock.

ON MODEL II :

In analogy to model I, the optimal control function takes on the values zero or one only, depending on whether the sign of $(1 - \psi_1)$ and $[(1-\lambda)Fk^{\alpha} - \pi k]$ will or will not agree. By a detailed discussion of these two functions, within the framework of the Pontryagin system, we can determine the optimal control function.

ON MODEL III :

The determination of the optimal solution is based on

$$u(\psi_1 - \psi_2)[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2]$$

$$= \sup_{\bar{u} \in U} \{\bar{u}(\psi_1 - \psi_2)[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2]\}.$$

The optimal control function u(t) thus takes on the values zero or one only, depending on whether the sign of $(\psi_1 - \psi_2)[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2)$ is negative or positive.

For a positive initial available product

$$(\gamma_1 - \pi)k_{10} - (\mu\gamma_2 + \pi)k_{20} > 0$$

for all $t \in [0,T]$ holds

$$[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2] > 0.$$

Therefore the optimal control function becomes zero for $\psi_1 < \psi_2$ and one for $\psi_1 > \psi_2$. Since the functions $\psi 1$ and $\psi 2$ are not equal in any interval and u(t) is piecewise continuous by definition, there exist only a finite number of points for which $\psi_1 = \psi_2$. holds. In this case, u(t) can be completed by one of its limits. The optimal control function u(t) is determined successively starting from t=T. There are again two possibilities:

$u(t) \equiv 0$ for t in a neighbourhood U(t) of $t = T$

$u(t) \equiv 1$ for t in a neighbourhood U(t) of $t = T$

It can be proved for this model, that the optimal control function has in the utmost two jump points in (0,T).

ON MODEL IV :

In analogy to model III, the optimal control function u(t) takes on the values zero or one only, depending on whether the sign of $(\psi_1 - \psi_2)[(\gamma_1 - \pi)k_1 - (\mu\gamma_2 + \pi)k_2]$ is negative or positive. For the time minimum problem of this model, however, there exists at the most one jump point. This is obvious, since a fixed planning period does not exist.

18. SUMMARY

The models considered here, deal with the concepts of optimal economic planning on the basis of the theory of controllable processes as developed by Pontryagin and his co-workers.[1] This analytical technique frequently was used for the formulation and solution of economic problems, primarily in the theory of economic growth.[2] This theory analyses so-called optimal steady state paths like the famous «golden rule» path and, from another point of view, the von Neumann path. The problem consisted in finding paths adapting to the mentioned paths, the so-called turnpikes, starting from an arbitrary intitial state.[3]

The first applications of the theory of controllable processes in economic theory consisted in the determination of such turnpikes. Kurz[4] discussed the adaption to a von Neumann path as a minimum time problem in a two sector model with neoclassical production functions. Stoleru[5] considered a slightly different problem, taking into regard constraints such as the maintenance of a minimum per

1. Pontryagin
2. Chakravarty
3. Dorfman, Samuelson, Solow
4. Kurz
5. Stoleru

capita consumption of foreign economic aid. He demonstrates his model with the help of statistical data of Algeria. Different versions of these models were discussed by Cass[1] and Shell.[2] Cass used a utility function in his model and Shell introduced lower boundaries for the final capital stocks.

However, some models only discuss the problem of an optimal steady state path with an infinite time horizon.[3]

If the objective function is an improper integral, mathematical problems arise.[4]

In the models mentioned above, the problem was the control of income-spending resp. allocation of the investments in multisectoral models. As an example for the allocation problem the reader is referred to the model of Takayama.[5] He discusses the optimal regional allocation of investments. In other types of models, the problem is the control of the income distribution.[6]

Finally, Arrow[7] used the theory of optimal control for the determination of an optimal

---

1. Cass
2. Shell
3. Uzawa [1]
4. Britsch, Reichardt, Schips [1]
5. Takayama
6. Britsch, Reichardt, Schips [2], Hamada
7. Arrow

investment policy in the theory of the firm.

The problems of the models discussed here are the following: An initial state and a wanted final state are described by capital stocks, and the problem is how to transform the economy in an optimal manner from the initial to the final state. The different alternatives, from which an optimal one is chosen, are different plans for the capital accumulation. However this is no decisive restriction, since our consideration can be easily extended to questions referring to problems of optimal education and regional development. The reader may consult, for example, a model by Uzawa.[1]

The main purpose of our models was to demonstrate the mathematical solutions in a way suitable for numerical illustrations. In the next steps we would have to develop numerical methods for the solution of more complicated models, which we would need in practical situations. Further problems lie in the estimation of the parameters of these models. The purpose of this lecture was mainly to show the possibilities of such models and to stimulate further research in this field.

---

1. Uzawa [2]

19. REFERENCES

K. J. Arrow (S. 59) Optimal Capital Policy with Irreversible Investment, J.N. Wolfe, ed., Value, Capital and Growth. Papers in Honour of Sir John Hicks, Edinburgh 1968, pp. 1

K. Britsch, H. Reichardt, B. Schips [1] (S. 59) Zur Optimalität der Uzawa-Pfade, Weltwirtschaftliches Archiv

K. Britsch, K. Reichardt, B. Schips [2] (S. 59) Sind die Lohnempfänger gut beraten, wenn sie sich einer Lohnquotensenkung widersetzen? Jahrbücher für Nationalökonomie und Statistik 183, 1969

D. Cass (S. 58) Optimum Growth in an Aggeregative Model of Capital Accumulation: A Turnpike Theorem, Econometrica 34, 1966, pp. 833

S. Chakravarty (S. 58) Optimal Programmes of Accumulation in an Multisector Economy, Econometrica, Vol. 33, pp. 557

R. Dorfman, P. Samuelson, R. M. Solow (S. 58) Linear Programming in Economic Analysis, New York, Toronto, London 1958

L. T. Fan (S. 12) The Continuous Maximum Principle, New York 1966

R. Frisch (S. 3) Statikk og dynamikk i den Økonomiske teori, Nationalokonomik Tidsskrift, 1929

R. M. Goodwin (S. 2) The Optimal Growth Path for an Underdeveloped Economy, Economic Journal 71, 1961

K. Hamada (S. 59) On the Optimal Transfer and Income Distribution in a Growing Economy, Review of Economic Studies 34, 1967, pp. 225

R. F. Harrod (S. 4) An Essay in Dynamic Theory, Economic Journal 49, 1939

M. Kurz (S. 1) (S. 58) Optimal Paths of Capital Accumulation under the Minimum Time Objective, Econometrica 33, 1965, pp. 42

G. Leitmann (S. 12) Optimization Techniques, New York 1962

L.S. Pontryagin (S. 11) Mathematische Theorie optimaler
V.G. Boltyanskiy (S. 12) Prozesse
R.V. Gamkrelidze (S. 15) München 1964.
E.F. Misçenko (S. 23)
(S. 29)
(S. 46)
(S. 58)

F.P. Ramsey (S. 2) A Mathematical Theory of Saving, Economic Journal 38, 1928

P.A. Samuelson (S. 2) A Catenary Turnpike Theorem Involving Consumption and the Golden Rule. American Economic Review 55, 1965

K. Shell (S. 58) Optimal Programmes of Capital Accumulation for an Economy in which there is Exogenous Technical Change, K. Shell, ed., Essays on the Theory of Optimal Economic Growth, Cambridge 1967, pp. 1

L. Stoleru (S. 42) (S. 58) An Optimal Policy for Economic Growth, Econometrica 33, 1965, pp. 321

A. Takayama (S. 59) Regional Allocation of Investment: A Further Analysis, Quarterly Journal of Economics 81, 1967, pp. 330

J. Tinbergen [1] (S. 2) Optimum Savings and Utility Maximization over Time, Econometrica 28, 1960.

J. Tinbergen [2] (S. 3) Econometric Business Cycle Research, Review of Economic Studies, Vol. VII, 1939/40, pp. 74, 75

J. Tinbergen [3] (S. 6) On the Theory of Economic Policy, Amsterdam 1963, pp. 6

J. Tinbergen [4] (S. 7) Economic Policy: Principles and Design, Amsterdam 1964

H. Uzawa [1] (S. 58) Optimal Growth in a Two Sector Model of Capital Accumulation, Review of Economic Studies 31, 1964, pp. 1

H. Uzawa [2] (S. 59) Optimum Technical Change in an Aggregative Model of Economic Growth, International Economic Review 6, 1965, pp. 18