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Forecasting quarterly GDP using a system of stochastic differential equations

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ABSTRACT

A new method for forecasting annual flow macroeconomic aggregates using monthly indicators is developed. A continuous time dynamic model is employed in order to obtain the exact discrete error correction model. The FIML estimation method is applied on the exact discrete error correction model to derive parameter estimates that are invariant to the sampling interval. The monthly indicators can be either I(0) or I(1). In the last case the system error correction format has an advantage in the estimation by encompassing the case cointegration.

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1. INTRODUCTION

The aim of this work is to construct a high frequency indicator for the state of Greek economy that best forecast economic activity using annual GDP as a benchmark. Forecasting GDP at quarterly or even monthly intervals is particularly important in the case of Greece since, up to now, there are no reliable official measures for quarterly GDP. To realize this task we utilize information from monthly time series coming from various sectors of the economy, both demand and supply side: e.g. manufacturing, construction, services, external trade.

Our forecasts are coming from a new technique based on the Exact Discrete Model (EDM) for a general observation interval h derived from an underlying continuous time dynamic Data Generating Process (DGP). We specify the DGP as a system of linear stochastic differential equations driven by a mixture of I(1) and I(0) forcing variables that Grange – cause the GDP time series. To implement the method initially we bring the EDM in an Error Correction Format (EDECM) and we estimate it by the Maximum Likelihood (ML) method. The Error Correction Format has the advantage to encompass the case of cointegration between the GDP and the related series while the EDM bring estimates which are unbiased and structural invariant to the observation interval h. For a survey on the estimation of stochastic differential equations see Bergrstrom [1984].

In section 2 we set the assumptions and we prove the theorem for the EDECM which provide the basis for the estimation methods. In section 3 we state the two methods for obtaining unbiased ML estimates for the monthly GDP. In parts A to D of the Mathematical Appendix we state the necessary technical facts that support the proof of the theorem in section 2. Finally in section 4 we conclude with some remarks.

2. THEOREM

Let the real continuous time random process $[y(t), z_1(t), z_2(t), t > 0]$ be a solution of the stochastic differential equation system

$$\begin{aligned} dy(t) &= c_1 + \alpha y(t) + \beta_1 z_1(t) + \beta_2 z_2(t) + \varepsilon_1(t) \\ dz_1(t) &= c_2 + \gamma_1 z_1 dt + \varepsilon_{2t} \\ dz_2(t) &= c_3 + \varepsilon_{3t} \end{aligned} \tag{1a}$$

where $\varepsilon(t) = \begin{bmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \\ \varepsilon_3(t) \end{bmatrix}$ is a continuous time white noise with covariance matrix $E\varepsilon(t)\varepsilon'(t) = \Sigma$ and $\begin{bmatrix} y(0) \\ z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} y_0 \\ z_{10} \\ z_{20} \end{bmatrix}$ are the initial conditions. We also assume that the solution for $y(t)$ is only observable as a sequence of integrals of the form $y_t = \int_{t-h}^{t} y(s)ds$ (flow variable) while the rest variables $z_1(t), z_2(t)$ are observed at the end of the observation period, (stock variables).

Then under the previous assumptions the equally spaced observations $[y(rh), z_1(rh), z_2(rh)]$, where $r = 1,2,...$ is integer and h is the observation interval, satisfy the following Exact Discrete Error Correction Model (**EDECM**):

$$\begin{aligned} \begin{bmatrix} \Delta\int_{t-h}^{t} y(s)ds \\ \Delta z_1(t) \\ \Delta z_2(t) \end{bmatrix} &= A_1 \begin{bmatrix} \int_{t-2h}^{t-h} y(s)ds \\ z_1(t-h) \\ z_2(t-h) \end{bmatrix} + A_2 \begin{bmatrix} \Delta\int_{t-2h}^{t-h} y(s)ds \\ \Delta z_1(t-h) \\ \Delta z_2(t-h) \end{bmatrix} + A_3 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \\ &\Theta_1 \begin{bmatrix} \zeta_{\varepsilon_1 t} \\ \zeta_{\varepsilon_2 t} \\ \zeta_{\varepsilon_3 t} \end{bmatrix} + \Theta_2 \begin{bmatrix} \zeta_{\varepsilon_1 t-h} \\ \zeta_{\varepsilon_2 t-h} \\ \zeta_{\varepsilon_3 t-h} \end{bmatrix} + \Theta_3 \begin{bmatrix} \zeta_{\varepsilon_1 t-2h} \\ \zeta_{\varepsilon_2 t-2h} \\ \zeta_{\varepsilon_3 t-2h} \end{bmatrix} \end{aligned} \tag{1b}$$

where

$$A_1 = \begin{bmatrix} e^{ah} - 1 & 0 & \frac{\beta_2}{-a} h\left(1 - e^{ah}\right) \\ 0 & e^{\gamma_1 h} - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & \frac{\beta_1}{\gamma_1}\frac{e^{\gamma_1 h} - e^{ah}}{\gamma_1 - a} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} \frac{e^{ah}-1}{a} & -h\frac{\beta_1}{\gamma_1 a}\left(e^{ah}-1\right) & -\frac{\beta_2}{a}\left\{\frac{1}{2}h^2\left(1-e^{ah}\right)+\frac{h}{a}\left(1-e^{ah}\right)+h^2 e^{ah}\right\} \\ 0 & -\frac{1}{\gamma_1}\left(1-e^{\gamma_1 h}\right) & 0 \\ 0 & 0 & h \end{bmatrix}$$

$$\Theta_1 = \begin{bmatrix} K_1^{\frac{1}{2}} & \beta_1 K_2^{\frac{1}{2}} & \frac{\beta_2}{a}\left(K_1^{\frac{1}{2}} - K_4^{\frac{1}{2}}\right) \\ 0 & K_2^{\frac{1}{2}} & 0 \\ 0 & 0 & K_3^{\frac{1}{2}} \end{bmatrix}$$

$$\Theta_2 = \begin{bmatrix} K_1^{\frac{1}{2}}C_1(h,a) & \beta_1 K_2^{\frac{1}{2}}C_2(h,a,\gamma_1) & -\frac{\beta_2}{a}\left\{he^{ah} - \frac{1-e^{ah}}{-a}\right\}K_3^{\frac{1}{2}} - \frac{\beta_2}{a}K_4^{\frac{1}{2}}C_4(h) + \frac{\beta_2}{a}e^{ah}K_4^{\frac{1}{2}} + \frac{\beta_2}{a}K_1^{\frac{1}{2}}C_1(h,a) - \frac{\beta_2}{a}K_1^{\frac{1}{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Theta_3 = \begin{bmatrix} 0 & 0 & \frac{\beta_2}{a}e^{ah}K_4^{\frac{1}{2}}C_4(h) - \frac{\beta_2}{a}K_1^{\frac{1}{2}}C_1(h,a) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Proof of the theorem

In matrix form (1a) is

$$d\begin{bmatrix} z_1(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix} + \begin{bmatrix} \gamma_1 & 0 \\ \beta_1 & a \end{bmatrix}\begin{bmatrix} z_1(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} z_2(t) + \begin{bmatrix} \varepsilon_2(t) \\ \varepsilon_1(t) \end{bmatrix}$$

$$dz_2(t) = c_3 + \varepsilon_{3t} \tag{2}$$

Applying the integral operator $\int_{t-h}^{t}$ in system (2) we obtain

$$\begin{bmatrix} z_1(t) - z_1(t-h) \\ y(t) - y(t-h) \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix} h + \begin{bmatrix} \gamma_1 & 0 \\ \beta_1 & a \end{bmatrix}\begin{bmatrix} \int_{t-h}^{t} z_1(s)ds \\ \int_{t-h}^{t} y(s)ds \end{bmatrix} + \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix}\int_{t-h}^{t} z_2(s)ds + \begin{bmatrix} \int_{t-h}^{t} \varepsilon_2(s)ds \\ \int_{t-h}^{t} \varepsilon_1(s)ds \end{bmatrix}$$

$$z_2(t) - z_2(t-h) = c_3 h + \int_{t-h}^{t} \varepsilon_{3t}(s)ds \tag{3}$$

Moreover from (see proof in Mathematical Appendix: part A)

$$\begin{aligned} \begin{bmatrix} \int_{t-h}^{t} z_1(s)ds \\ \int_{t-h}^{t} y(s)ds \end{bmatrix} &= (e^{Ah} - I)A^{-1}\begin{bmatrix} c_2 \\ c_1 \end{bmatrix} h + e^{Ah}\begin{bmatrix} \int_{t-2h}^{t-h} z_1(s)ds \\ \int_{t-h}^{t} y(s)ds \end{bmatrix} + \\ & A^{-1}\int_{t-h}^{t}\left\{e^{A(t-s)} - I\right\}\begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} z_2(s)ds + A^{-1}\int_{t-2h}^{t-h}\left\{e^{Ah} - e^{A(t-h-s)}\right\}\begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} z_2(s)ds + \xi_t(h), \\ \xi_t(h) &= A^{-1}\int_{t-h}^{t}\left\{e^{A(t-s)} - I\right\}\begin{bmatrix} \varepsilon_2(s) \\ \varepsilon_1(s) \end{bmatrix} ds + A^{-1}\int_{t-2h}^{t-h}\left\{e^{Ah} - e^{A(t-h-s)}\right\}\begin{bmatrix} \varepsilon_2(s) \\ \varepsilon_1(s) \end{bmatrix} ds, \\ A &= \begin{bmatrix} \gamma_1 & 0 \\ \beta_1 & a \end{bmatrix} \end{aligned} \tag{4}$$

from the first equation of system (3)

$$z_1(t) - z_1(t-h) = c_2 h + \gamma_1\int_{t-h}^{t} z_1(s)ds + \int_{t-h}^{t} \varepsilon_2(s)ds \tag{5}$$

from (5)

$$\int_{t-h}^{t} z_1(s)ds = \frac{1}{\gamma_1}\left\{z_1(t) - z_1(t-h) - \left(c_2 h - \int_{t-h}^{t} \varepsilon_2(s)ds\right)\right\} \tag{6}$$

Now we substitute (6) in (4) to eliminate the unobservable $\int_{t-h}^{t} z_1(s)ds$:

$$\begin{bmatrix} \frac{1}{\gamma_1}\{z_1(t)-z_1(t-h)\} \\ \int_{t-h}^{t} y(s)ds \end{bmatrix} = \frac{1}{\gamma_1}\begin{bmatrix} c_2 h - \int_{t-h}^{t} \varepsilon_2(s)ds \\ 0 \end{bmatrix} + (e^{Ah}-I)A^{-1}\begin{bmatrix} c_2 \\ c_1 \end{bmatrix} h +$$

$$e^{Ah}\left(\begin{bmatrix} \frac{1}{\gamma_1}\{z_1(t-h)-z_1(t-2h)\} \\ \int_{t-h}^{t} y(s)ds \end{bmatrix} - \frac{1}{\gamma_1}\begin{bmatrix} c_2 h - \int_{t-2h}^{t-h} \varepsilon_2(s)ds \\ 0 \end{bmatrix} \right) +$$

$$A^{-1}\int_{t-h}^{t}\left\{e^{A(t-s)}-I\right\}\begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} z_2(s)ds + A^{-1}\int_{t-2h}^{t-h}\left\{e^{Ah}-e^{A(t-h-s)}\right\}\begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} z_2(s)ds +$$

$$\xi_t(h)$$

or

$$\begin{bmatrix} \frac{1}{\gamma_1}\{z_1(t)-z_1(t-h)\} \\ \int_{t-h}^{t} y(s)ds \end{bmatrix} = \frac{1}{\gamma_1}\begin{bmatrix} c_2 h - \int_{t-h}^{t} \varepsilon_2(s)ds \\ 0 \end{bmatrix}$$

$$(e^{Ah}-I)A^{-1}\begin{bmatrix} c_2 \\ c_1 \end{bmatrix} h + e^{Ah}\begin{bmatrix} \frac{1}{\gamma_1}\{z_1(t-h)-z_1(t-2h)\} \\ \int_{t-h}^{t} y(s)ds \end{bmatrix} - e^{Ah}\frac{1}{\gamma_1}\begin{bmatrix} c_2 h - \int_{t-2h}^{t-h} \varepsilon_2(s)ds \\ 0 \end{bmatrix} + \qquad (7)$$

$$A^{-1}\int_{t-h}^{t}\left\{e^{A(t-s)}-I\right\}\begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} z_2(s)ds + \int_{t-2h}^{t-h}\left\{A^{-1}e^{Ah}-A^{-1}e^{A(t-h-s)}\right\}\begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} z_2(s)ds + \xi_t(h)$$

notice that

$$A = \begin{bmatrix} A^{ss} & A^{sf} \\ A^{fs} & A^{ff} \end{bmatrix} = \begin{bmatrix} \gamma_1 & 0 \\ \beta_1 & a \end{bmatrix}, H(s) = e^{As} = \begin{bmatrix} e^{\gamma_1 s} & 0 \\ \beta_1 \frac{e^{\gamma_1 s}-e^{as}}{\gamma_1 - a} & e^{as} \end{bmatrix}, A^{-1} = \frac{1}{\gamma_1 a}\begin{bmatrix} \alpha & 0 \\ -\beta_1 & \gamma_1 \end{bmatrix}$$

therefore from (7)

$$\begin{bmatrix} \frac{1}{\gamma_1}\{z_1(t)-z_1(t-h)\} \\ \int_{t-h}^{t} y(s)ds \end{bmatrix} = \frac{1}{\gamma_1}\begin{bmatrix} c_2 h - \int_{t-h}^{t} \varepsilon_2(s)ds \\ 0 \end{bmatrix} + \frac{1}{\gamma_1 a}\begin{bmatrix} e^{\gamma_1 h}-1 & 0 \\ \beta_1 \frac{e^{\gamma_1 h}-e^{ah}}{\gamma_1 - a} & e^{ah}-1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ -\beta_1 & \gamma_1 \end{bmatrix}\begin{bmatrix} c_2 \\ c_1 \end{bmatrix} h +$$

$$\begin{bmatrix} e^{\gamma_1 h} & 0 \\ \beta_1 \frac{e^{\gamma_1 h}-e^{ah}}{\gamma_1 - a} & e^{ah} \end{bmatrix}\begin{bmatrix} \frac{1}{\gamma_1}\{z_1(t-h)-z_1(t-2h)\} \\ \int_{t-h}^{t} y(s)ds \end{bmatrix} - \frac{1}{\gamma_1}\begin{bmatrix} e^{\gamma_1 h} & 0 \\ \beta_1 \frac{e^{\gamma_1 h}-e^{ah}}{\gamma_1 - a} & e^{ah} \end{bmatrix}\begin{bmatrix} c_2 h - \int_{t-2h}^{t-h} \varepsilon_2(s)ds \\ 0 \end{bmatrix} +$$

$$\int_{t-h}^{t}\frac{1}{\gamma_1 a}\begin{bmatrix} \alpha & 0 \\ -\beta_1 & \gamma_1 \end{bmatrix}\begin{bmatrix} e^{\gamma_1(t-s)}-1 & 0 \\ \beta_1 \frac{e^{\gamma_1(t-s)}-e^{a(t-s)}}{\gamma_1 - a} & e^{a(t-s)}-1 \end{bmatrix}\begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} z_2(s)ds +$$

$$\frac{1}{\gamma_1 a}\begin{bmatrix} \alpha & 0 \\ -\beta_1 & \gamma_1 \end{bmatrix}\int_{t-2h}^{t-h}\left\{\begin{bmatrix} e^{\gamma_1 h} & 0 \\ \beta_1 \frac{e^{\gamma_1 h}-e^{ah}}{\gamma_1 - a} & e^{ah} \end{bmatrix} - \begin{bmatrix} e^{\gamma_1(t-h-s)} & 0 \\ \beta_1 \frac{e^{\gamma_1(t-h-s)}-e^{a(t-h-s)}}{\gamma_1 - a} & e^{a(t-h-s)} \end{bmatrix}\right\}\begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} z_2(s)ds +$$

$$\xi_t(h)$$

(8)

therefore from (8) we obtain

$$\begin{bmatrix} \frac{1}{\gamma_1}\{z_1(t)-z_1(t-h)\} \\ \int_{t-h}^{t} y(s)ds \end{bmatrix} = \frac{1}{\gamma_1}\begin{bmatrix} c_2 h - \int_{t-h}^{t}\varepsilon_2(s)ds \\ 0 \end{bmatrix} + h\begin{bmatrix} \frac{e^{\gamma_1 h}-1}{\gamma_1}c_2 \\ \left\{\frac{\beta_1}{\gamma_1}\frac{e^{\gamma_1 h}-e^{ah}}{\gamma_1 - a} - \frac{\beta_1}{\gamma_1 a}\left(e^{ah}-1\right)\right\}c_2 + \frac{e^{ah}-1}{a}c_1 \end{bmatrix} +$$

$$\begin{bmatrix} \frac{e^{\gamma_1 h}}{\gamma_1}\{z_1(t-h)-z_1(t-2h)\} \\ \frac{\beta_1}{\gamma_1}\frac{e^{\gamma_1 h}-e^{ah}}{\gamma_1 - a}\{z_1(t-h)-z_1(t-2h)\} + e^{ah}\int_{t-h}^{t} y(s)ds \end{bmatrix} - \begin{bmatrix} \frac{e^{\gamma_1 h}}{\gamma_1}\left\{c_2 h - \int_{t-2h}^{t-h}\varepsilon_2(s)ds\right\} \\ \frac{\beta_1}{\gamma_1}\frac{e^{\gamma_1 h}-e^{ah}}{\gamma_1 - a}\left\{c_2 h - \int_{t-2h}^{t-h}\varepsilon_2(s)ds\right\} \end{bmatrix} +$$

$$\int_{t-h}^{t}\begin{bmatrix} 0 \\ \beta_2\frac{e^{a(t-s)}-1}{a} \end{bmatrix} z_2(s)ds +$$

$$\int_{t-2h}^{t-h}\begin{bmatrix} \frac{e^{\gamma_1 h}-e^{\gamma_1(t-h-s)}}{\gamma_1} & 0 \\ -\frac{\beta_1}{\gamma_1 a}e^{\gamma_1 h} + \frac{\beta_1}{a}\frac{e^{\gamma_1 h}-e^{ah}}{\gamma_1 - a} + \frac{\beta_1}{\gamma_1 a}e^{\gamma_1(t-h-s)} - \frac{\beta_1}{a}\frac{e^{\gamma_1(t-h-s)}-e^{a(t-h-s)}}{\gamma_1 - a} & \frac{e^{ah}}{a} - \frac{e^{a(t-h-s)}}{a} \end{bmatrix}\begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} z_2(s)ds +$$

$$\xi_t(h)$$

(9)

From (9) we obtain

$$\begin{aligned} \int_{t-h}^{t} y(s)ds &= h\left\{\frac{\beta_1}{\gamma_1}\frac{e^{\gamma_1 h}-e^{ah}}{\gamma_1 - a} - \frac{\beta_1}{\gamma_1 a}\left(e^{ah}-1\right)\right\}c_2 + h\frac{e^{ah}-1}{a}c_1 + \\ &\frac{\beta_1}{\gamma_1}\frac{e^{\gamma_1 h}-e^{ah}}{\gamma_1 - a}\{z_1(t-h)-z_1(t-2h)\} + e^{ah}\int_{t-2h}^{t-h} y(s)ds - \\ &\frac{\beta_1}{\gamma_1}\frac{e^{\gamma_1 h}-e^{ah}}{\gamma_1 - a}\left\{c_2 h - \int_{t-2h}^{t-h}\varepsilon_2(s)ds\right\} + \\ &\frac{\beta_2}{a}\int_{t-h}^{t}\left\{e^{a(t-s)}-1\right\}z_2(s)ds + \frac{\beta_2}{a}\int_{t-2h}^{t-h}\left\{e^{ah}-e^{a(t-h-s)}\right\}z_2(s)ds + \\ &\xi_{2t}(h) \end{aligned} \tag{10}$$

An expression for the integral of $z_2(s)$ in (10) can be found in Appendix part B:

$$\begin{aligned} &\frac{\beta_2}{a}\int_{t-h}^{t}\left\{e^{a(t-s)}-1\right\}z_2(s)ds + \frac{\beta_2}{a}\int_{t-2h}^{t-h}\left\{e^{ah}-e^{a(t-h-s)}\right\}z_2(s)ds = \\ &z_2(t-h)\frac{\beta_2}{-a}h\left(1-e^{ah}\right) + c_3\frac{\beta_2}{-a}\left\{\frac{1}{2}h^2\left(1-e^{ah}\right) - \frac{h}{-a}\left(1-e^{ah}\right) + h^2 e^{ah}\right\} \\ &-\frac{\beta_2}{a}\left\{he^{ah} - \frac{1-e^{ah}}{-a}\right\}\int_{t-2h}^{t-h}\varepsilon_{3t}(s)ds + \vartheta_t \end{aligned} \tag{11}$$

Finally substituting (11) in (10) we obtain

$$\int_{t-h}^{t} y(s)ds = e^{ah}\int_{t-2h}^{t-h} y(s)ds + \frac{\beta_1}{\gamma_1}\frac{e^{\gamma_1 h} - e^{ah}}{\gamma_1 - a}\{z_1(t-h) - z_1(t-2h)\} + z_2(t-h)\frac{\beta_2}{-a}h\left(1-e^{ah}\right)$$
$$+ h\frac{e^{ah}-1}{a}c_1 - h\frac{\beta_1}{\gamma_1 a}\left(e^{ah}-1\right)c_2 - \frac{\beta_2}{a}\left\{\frac{1}{2}h^2\left(1-e^{ah}\right) + \frac{h}{a}\left(1-e^{ah}\right) + h^2 e^{ah}\right\}c_3 + \lambda_{1t}(h) \tag{12}$$

where

$$\lambda_{1t}(h) = -\frac{\beta_2}{a}\left\{he^{ah} - \frac{1-e^{ah}}{-a}\right\}\int_{t-2h}^{t-h}\varepsilon_{3t}(s)ds + \frac{\beta_1}{\gamma_1}\frac{e^{\gamma_1 h}-e^{ah}}{\gamma_1 - a}\int_{t-2h}^{t-h}\varepsilon_2(s)ds + \xi_{2t}(h) + \vartheta_t \tag{13}$$

while using the expression for $\xi_{2t}(h)$ from Appendix C we obtain:

$$\begin{aligned}
&\lambda_{1t}(h) = \\
&\int_{t-h}^{t}\frac{1}{-a}\left(1-e^{a(t-s)}\right)\varepsilon_1(s) + \int_{t-2h}^{t-h}\frac{1}{a}\left(e^{ah} - e^{a(t-h-s)}\right)\varepsilon_1(s) \\
&\beta_1\int_{t-h}^{t}\left\{\frac{1}{\gamma_1 a}\left(1-e^{\gamma_1(t-s)}\right) + \frac{1}{a}\frac{e^{\gamma_1(t-s)} - e^{a(t-s)}}{\gamma_1 - a}\right\}\varepsilon_2(s) + \\
&\beta_1\int_{t-2h}^{t-h}\left\{-\frac{1}{\gamma_1 a}\left(e^{\gamma_1 h} - e^{\gamma_1(t-h-s)}\right) + \frac{1}{\gamma_1}\frac{e^{\gamma_1 h}-e^{ah}}{\gamma_1 - a} + \frac{1}{a}\left(\frac{e^{\gamma_1 h}-e^{ah}}{\gamma_1 - a} - \frac{e^{\gamma_1(t-h-s)} - e^{a(t-h-s)}}{\gamma_1 - a}\right)\right\}\varepsilon_2(s) \\
&-\frac{\beta_2}{a}\left\{he^{ah} - \frac{1-e^{ah}}{-a}\right\}\int_{t-2h}^{t-h}\varepsilon_{3t}(s)ds \\
&-\frac{\beta_2}{a}\int_{t-h}^{t}\int_{s-h}^{s}\varepsilon_3(\theta)d\theta ds + \frac{\beta_2}{a}e^{ah}\int_{t-2h}^{t-h}\int_{s-2h}^{s-h}\varepsilon_3(\theta)d\theta ds \\
&+\frac{\beta_2}{a}\int_{t-h}^{t}\int_{s-h}^{s}e^{a(t-s)}\varepsilon_3(\theta)d\theta ds - \frac{\beta_2}{a}\int_{t-2h}^{t-h}\int_{s-2h}^{s-h}e^{a(t-h-s)}\varepsilon_3(\theta)d\theta ds
\end{aligned} \tag{14}$$

From the exact discrete model of the last two equations of (1) (see also Bergstrom [1984]) we obtain:

$$z_1(t) = e^{\gamma_1 h}z_1(t-h) + \frac{c_2}{-\gamma_1}\left[1-e^{\gamma_1 h}\right] + \lambda_{2t}(h) \tag{15}$$

$$z_2(t) = z_2(t-h) + c_3 h + \lambda_{3t}(h) \tag{16}$$

where

$$\begin{aligned}
&\lambda_{2t}(h) = \int_{t-h}^{t}e^{\gamma_1(t-s)}\varepsilon_2(s)ds = K_2^{\frac{1}{2}}\zeta_{\varepsilon_{2t}},\ \lambda_{3t}(h) = \int_{t-h}^{t}\varepsilon_{3t}(s)ds = K_3^{\frac{1}{2}}\zeta_{\varepsilon_{3t}} \\
&K_2 = \int_{t-h}^{t}e^{2\gamma_1(t-s)}ds = \frac{1}{-2\gamma_1}\left[1-e^{2\gamma_1 h}\right], K_3 = \int_{t-h}^{t}ds = h \\
&\zeta_{\varepsilon_{3t}} \to N(0,\sigma_{\varepsilon_3}^2), \zeta_{\varepsilon_{2t}} \to N(0,\sigma_{\varepsilon_2}^2)
\end{aligned} \tag{17}$$

Moreover notice that:

$$\int_{t-h}^{t}\int_{s-h}^{s} e^{a(t-s)}\varepsilon_3(\theta)d\theta ds = \int_{t-h}^{t}\left[\int_{\theta}^{t} e^{a(t-s)}ds\right]\varepsilon_3(\theta)d\theta + \int_{t-2h}^{t-h}\left[\int_{t-h}^{\theta+h} e^{a(t-s)}ds\right]\varepsilon_3(\theta)d\theta =$$

$$\frac{1}{a}\int_{t-h}^{t}\left[e^{a(t-\theta)}-1\right]\varepsilon_3(\theta)d\theta + \frac{1}{a}\int_{t-2h}^{t-h}\left[e^{ah}-e^{a(t-h-\theta)}\right]\varepsilon_3(\theta)d\theta = K_1^{\frac{1}{2}}\left\{\zeta_{\varepsilon_3 t} + C_1(h,a)\zeta_{\varepsilon_3 t-h}\right\}$$

$$\zeta_{\varepsilon_3 t} \to N(0,\sigma^2{}_{\varepsilon_3})$$

and similarly

$$\int_{t-2h}^{t-h}\int_{s-2h}^{s-h} e^{a(t-h-s)}\varepsilon_3(\theta)d\theta ds = \frac{1}{a}\int_{t-2h}^{t-h}\left[e^{a(t-h-\theta)}-1\right]\varepsilon_3(\theta)d\theta + \frac{1}{a}\int_{t-3h}^{t-2h}\left[e^{ah}-e^{a(t-2h-\theta)}\right]\varepsilon_3(\theta)d\theta =$$

$$K_1^{\frac{1}{2}}\left\{\zeta_{\varepsilon_3 t-h} + C_1(h,a)\zeta_{\varepsilon_3 t-2h}\right\}$$

moreover

$$\int_{t-h}^{t}\int_{s-h}^{s}\varepsilon_3(\theta)d\theta ds = \int_{t-h}^{t}\left[\int_{\theta}^{t} ds\right]\varepsilon_3(\theta)d\theta + \int_{t-2h}^{t-h}\left[\int_{t-h}^{\theta+h} ds\right]\varepsilon_3(\theta)d\theta =$$

$$\int_{t-h}^{t}(t-\theta)\varepsilon_3(\theta)d\theta + \int_{t-2h}^{t-h}(\theta+h-t+h)\varepsilon_3(\theta)d\theta = K_4^{\frac{1}{2}}\left\{\zeta_{\varepsilon_3 t} + C_4(h)\zeta_{\varepsilon_3 t-h}\right\}$$

Therefore we have

$$\lambda_{1t}(h) = K_1^{\frac{1}{2}}\left\{\zeta_{\varepsilon_1 t} + C_1(h,a)\zeta_{\varepsilon_1 t-h}\right\} + \beta_1 K_2^{\frac{1}{2}}\left[\zeta_{\varepsilon_2 t} + C_2(h,a,\gamma_1)\zeta_{\varepsilon_2 t-h}\right] - \frac{\beta_2}{a}\left\{he^{ah} - \frac{1-e^{ah}}{-a}\right\}K_3^{\frac{1}{2}}\zeta_{\varepsilon_{3t-h}}$$

$$-\frac{\beta_2}{a}K_4^{\frac{1}{2}}\left\{\zeta_{\varepsilon_3 t} + C_4(h)\zeta_{\varepsilon_3 t-h}\right\} + \frac{\beta_2}{a}e^{ah}K_4^{\frac{1}{2}}\left\{\zeta_{\varepsilon_3 t-h} + C_4(h)\zeta_{\varepsilon_3 t-2h}\right\} + \frac{\beta_2}{a}K_1^{\frac{1}{2}}\left\{\zeta_{\varepsilon_3 t} + C_1(h,a)\zeta_{\varepsilon_3 t-h}\right\} -$$

$$\frac{\beta_2}{a}K_1^{\frac{1}{2}}\left\{\zeta_{\varepsilon_3 t-h} + C_1(h,a)\zeta_{\varepsilon_3 t-2h}\right\}$$

or

$$\lambda_{1t}(h) = K_1^{\frac{1}{2}}\left\{\zeta_{\varepsilon_1 t} + C_1(h,a)\zeta_{\varepsilon_1 t-h}\right\} + \beta_1 K_2^{\frac{1}{2}}\left[\zeta_{\varepsilon_2 t} + C_2(h,a,\gamma_1)\zeta_{\varepsilon_2 t-h}\right] - \frac{\beta_2}{a}\left\{he^{ah} - \frac{1-e^{ah}}{-a}\right\}K_3^{\frac{1}{2}}\zeta_{\varepsilon_{3t-h}}$$

$$+\frac{\beta_2}{a}K_4^{\frac{1}{2}}\left\{\left(\zeta_{\varepsilon_3 t-h}e^{ah} - \zeta_{\varepsilon_3 t}\right) + C_4(h)\left(\zeta_{\varepsilon_3 t-2h}e^{ah} - \zeta_{\varepsilon_3 t-h}\right)\right\} + \frac{\beta_2}{a}K_1^{\frac{1}{2}}\left\{\Delta\zeta_{\varepsilon_3 t} + C_1(h,a)\Delta\zeta_{\varepsilon_3 t-h}\right\} \tag{18}$$

From (17) and (18) in matrix form we obtain

$$\begin{bmatrix}\lambda_{1t}(h)\\ \lambda_{2t}(h)\\ \lambda_{3t}(h)\end{bmatrix} = \Theta_1\begin{bmatrix}\zeta_{\varepsilon_1 t}\\ \zeta_{\varepsilon_2 t}\\ \zeta_{\varepsilon_3 t}\end{bmatrix} + \Theta_2\begin{bmatrix}\zeta_{\varepsilon_1 t-h}\\ \zeta_{\varepsilon_2 t-h}\\ \zeta_{\varepsilon_3 t-h}\end{bmatrix} + \Theta_3\begin{bmatrix}\zeta_{\varepsilon_1 t-2h}\\ \zeta_{\varepsilon_2 t-2h}\\ \zeta_{\varepsilon_3 t-2h}\end{bmatrix} \tag{19}$$

where

$$\Theta_1 = \begin{bmatrix} K_1^{\frac{1}{2}} & \beta_1 K_2^{\frac{1}{2}} & \frac{\beta_2}{a}\left(K_1^{\frac{1}{2}} - K_4^{\frac{1}{2}} \right) \\ 0 & K_2^{\frac{1}{2}} & 0 \\ 0 & 0 & K_3^{\frac{1}{2}} \end{bmatrix}$$

$$\Theta_2 = \begin{bmatrix} K_1^{\frac{1}{2}} C_1(h,a) & \beta_1 K_2^{\frac{1}{2}} C_2(h,a,\gamma_1) & -\frac{\beta_2}{a}\left\{ he^{ah} - \frac{1-e^{ah}}{-a} \right\} K_3^{\frac{1}{2}} - \frac{\beta_2}{a} K_4^{\frac{1}{2}} C_4(h) + \frac{\beta_2}{a} e^{ah} K_4^{\frac{1}{2}} + \frac{\beta_2}{a} K_1^{\frac{1}{2}} C_1(h,a) - \frac{\beta_2}{a} K_1^{\frac{1}{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Theta_3 = \begin{bmatrix} 0 & 0 & \frac{\beta_2}{a} e^{ah} K_4^{\frac{1}{2}} C_4(h) - \frac{\beta_2}{a} K_1^{\frac{1}{2}} C_1(h,a) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From equations (12), (15), (16) and (19) we obtain the system in error correction format EDECM (1b).

3. ESTIMATION BY MAXIMUM LIKELIHOOD

To obtain the maximum likelihood parameter estimates we maximize

$$\log L = -\frac{T}{2}\log(\det\Sigma) - \frac{1}{2}\sum_{r=1h}^{Th}\left[\zeta_{\varepsilon_1 t} \quad \zeta_{\varepsilon_2 t} \quad \zeta_{\varepsilon_3 t}\right]\Sigma^{-1}\begin{bmatrix}\zeta_{\varepsilon_1 t}\\ \zeta_{\varepsilon_2 t}\\ \zeta_{\varepsilon_3 t}\end{bmatrix}$$

where $\left[\zeta_{\varepsilon_1 t} \quad \zeta_{\varepsilon_2 t} \quad \zeta_{\varepsilon_3 t}\right]'$, the innovations, are derived by the following recursion:

Initial value

$$\begin{bmatrix}\zeta_{\varepsilon_1 0}\\ \zeta_{\varepsilon_2 0}\\ \zeta_{\varepsilon_3 0}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

for $t = h$

$$\zeta_{\varepsilon_{1h}} =$$

$$\frac{a}{\pi_1}\left[\int_0^h y(s) - \left\{\frac{1}{a}\left(e^{a(h)}-1\right)y(0) + \left(\frac{1}{a}\beta_1\frac{e^{a(h)}-e^{\gamma(h)}}{a-\gamma} - \beta_1\frac{1}{a\gamma}\left(e^{\gamma(h)}-1\right)\right)z_1(0) + z_2(0)\frac{\beta_2}{a}\left(\frac{1-e^{ah}}{-a}-h\right)\right.\right.$$

$$-\left(\frac{1}{a}\left(\frac{1}{a}\left(1-e^{a(h)}\right)+h\right)c_1 + \left(-\frac{\beta_1}{a^2}\frac{e^{a(h)}-e^{\gamma(h)}}{a-\gamma} + \beta_1\left(e^{\gamma(h)}-1\right)\frac{a+\gamma}{a^2\gamma^2} - \beta_1\frac{1}{a\gamma}h\right)c_2\right) +$$

$$\left.\left. + c_3\frac{1}{-a}\left[h - \frac{1-e^{ah}}{-a}\right]\frac{\beta_2}{a} - \frac{\beta_2}{a}\frac{1}{2}c_3h^2 + \frac{\beta_1}{a}\pi_2\zeta_{\varepsilon_{2h}} + \frac{\beta_2}{a}\pi_3\zeta_{\varepsilon_{3h}} - \frac{\beta_2}{a}\pi_4\zeta_{\varepsilon_{3h}}\right\}\right]$$

and

$$\zeta_{\varepsilon_{2h}} = \frac{1}{\pi_5}\left[z_1(h) - \left\{e^{\gamma_1 h}z_1(0) + \frac{c_2}{-\gamma_1}\left[1-e^{\gamma_1 h}\right]\right\}\right]$$

$$\zeta_{\varepsilon_{3h}} = \frac{1}{\pi_6}\left[z_2(h) - \{z_2(0) + c_3h\}\right]$$

the first equation is derived by solving (D9) and the last two from the exact discrete model of (1a).

For $t = 2h$ we obtain

$$\begin{bmatrix} \zeta_{\varepsilon_1 2h} \\ \zeta_{\varepsilon_2 2h} \\ \zeta_{\varepsilon_3 2h} \end{bmatrix} = \Theta_1^{-1} \left\{ \begin{bmatrix} \Delta \int_h^{2h} y(s) ds \\ \Delta z_1(2h) \\ \Delta z_2(2h) \end{bmatrix} - \left\{ A_1 \begin{bmatrix} \int_0^h y(s) ds \\ z_1(h) \\ z_2(h) \end{bmatrix} + A_2 \begin{bmatrix} \Delta \int_0^h y(s) ds \\ \Delta z_1(h) \\ \Delta z_2(h) \end{bmatrix} + A_3 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \Theta_2 \begin{bmatrix} \zeta_{\varepsilon_1 h} \\ \zeta_{\varepsilon_2 h} \\ \zeta_{\varepsilon_3 h} \end{bmatrix} + \Theta_3 \begin{bmatrix} \zeta_{\varepsilon_1 0} \\ \zeta_{\varepsilon_2 0} \\ \zeta_{\varepsilon_3 0} \end{bmatrix} \right\} \right\}$$

For $t = rh$, $r = 3....$ we solve recursively

$$\begin{bmatrix} \zeta_{\varepsilon_1 t} \\ \zeta_{\varepsilon_2 t} \\ \zeta_{\varepsilon_3 t} \end{bmatrix} = \Theta_1^{-1} \left\{ \begin{bmatrix} \Delta \int_{t-h}^{t} y(s) ds \\ \Delta z_1(t) \\ \Delta z_2(t) \end{bmatrix} - \left\{ A_1 \begin{bmatrix} \int_{t-2h}^{t-h} y(s) ds \\ z_1(t-h) \\ z_2(t-h) \end{bmatrix} + A_2 \begin{bmatrix} \Delta \int_{t-2h}^{t-h} y(s) ds \\ \Delta z_1(t-h) \\ \Delta z_2(t-h) \end{bmatrix} + A_3 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \Theta_2 \begin{bmatrix} \zeta_{\varepsilon_1 t-h} \\ \zeta_{\varepsilon_2 t-h} \\ \zeta_{\varepsilon_3 t-h} \end{bmatrix} + \Theta_3 \begin{bmatrix} \zeta_{\varepsilon_1 t-2h} \\ \zeta_{\varepsilon_2 t-2h} \\ \zeta_{\varepsilon_3 t-2h} \end{bmatrix} \right\} \right\}$$

Estimation methods based on the likelihood function

A. Iterated ML

Steps:

1. Maximize the log-likelihood setting the observation step h=1 to derive unbiased parameters estimates.
2. Simulate the model to find $y(rh),\ r = 1...$ setting h=1/12 for the monthly observation interval.
3. Re-estimate the model using the values $y(rh),\ r = 1...$ from the previous step and the monthly observations of the related series to find new estimates for $y(rh),\ r = 1...$.
4. The procedure in step 3 can be iterated until convergence.

B. Skipping method

Steps:

1. Run recursively the EDECM for h=1/12 12-month periods back.
2. Sum $\sum_{j=1}^{12} (\ )_{t-j} \equiv \bar{y}_t$ both sides of the equations to find a system in terms of the observable variables

Solve the derived system from step 2 with respect to $\zeta_t = [\zeta_{1t} \quad \zeta_{2t} \quad \zeta_{3t}]$ and run recursively until you find $\zeta_t, \zeta_{t-12}, \zeta_{t-24}, ...$ in terms of the observable non-overlapping annual $\bar{y}_t$ and the monthly x_t. Construct the likelihood function skipping the unobservable $\bar{y}_{t-h}$.

3. CONCLUSION

This paper derives the Exact Discrete Model from an underlying continuous time model that can be applied to estimate econometric relations on variables with different observation interval. The EDM has the advantage of delivering unbiased estimates independently of the observation interval in contrast to more traditional time series models (e.g arma) and various approximations of continuous time models.

Further work is in progress for extending the class of models under consideration.

MATHEMATICAL APPENDIX

PART A

Proof of the Exact Discrete Model (4)

$$d\begin{bmatrix} z_1(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix} + \begin{bmatrix} \gamma_1 & 0 \\ \beta_1 & a \end{bmatrix}\begin{bmatrix} z_1(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} z_2(t) + \begin{bmatrix} \varepsilon_2(t) \\ \varepsilon_1(t) \end{bmatrix} \tag{A1}$$

from (3) we obtain

$$\begin{bmatrix} \int_{t-h}^{t} z_1(s)ds \\ \int_{t-h}^{t} y(s)ds \end{bmatrix} =$$
$$\begin{bmatrix} \gamma_1 & 0 \\ \beta_1 & a \end{bmatrix}^{-1}\left\{\begin{bmatrix} z_1(t) - z_1(t-h) \\ y(t) - y(t-h) \end{bmatrix} - \left(\begin{bmatrix} c_2 \\ c_1 \end{bmatrix} h + \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix}\int_{t-h}^{t} z_2(s)ds + \begin{bmatrix} \int_{t-h}^{t} \varepsilon_2(s)ds \\ \int_{t-h}^{t} \varepsilon_1(s)ds \end{bmatrix}\right)\right\} \tag{A2}$$

also from (A1)

$$\begin{bmatrix} z_1(t) \\ y(t) \end{bmatrix} = e^{Ah}\begin{bmatrix} z_1(t-h) \\ y(t-h) \end{bmatrix} + \int_{t-h}^{t} e^{A(t-s)}\left\{\begin{bmatrix} c_2 \\ c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} z_2(s) + \begin{bmatrix} \varepsilon_2(s) \\ \varepsilon_1(s) \end{bmatrix}\right\} ds \tag{A3}$$

apply the operator Δ_h in (A3) to obtain

$$\Delta_h\begin{bmatrix} z_1(t) \\ y(t) \end{bmatrix} = e^{Ah}\Delta_h\begin{bmatrix} z_1(t-h) \\ y(t-h) \end{bmatrix} + \Delta_h\int_{t-h}^{t} e^{A(t-s)}\left\{\begin{bmatrix} c_2 \\ c_1 \end{bmatrix} + \begin{bmatrix} \mu_2 \\ \mu_1 \end{bmatrix} s + \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} z_2(s) + \begin{bmatrix} \varepsilon_2(s) \\ \varepsilon_1(s) \end{bmatrix}\right\} ds \tag{A4}$$

substituting (A4) in (A2) we obtain

$$\begin{bmatrix} \int_{t-h}^{t} z_1(s)ds \\ \int_{t-h}^{t} y(s)ds \end{bmatrix} = A^{-1}\left\{e^{Ah}\Delta_h\begin{bmatrix} z_1(t-h) \\ y(t-h) \end{bmatrix} + \Delta_h\int_{t-h}^{t} e^{A(t-s)}\left\{\begin{bmatrix} c_2 \\ c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} z_2(s) + \begin{bmatrix} \varepsilon_1(s) \\ \varepsilon_2(s) \end{bmatrix}\right\} ds\right\} -$$
$$A^{-1}\left\{\begin{bmatrix} c_2 \\ c_1 \end{bmatrix} h + \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix}\int_{t-h}^{t} z_2(s)ds + \begin{bmatrix} \int_{t-h}^{t} \varepsilon_2(s)ds \\ \int_{t-h}^{t} \varepsilon_1(s)ds \end{bmatrix}\right\} \tag{A5}$$

lagging relation (3) by h we obtain

$$\Delta_h\begin{bmatrix} z_1(t-h) \\ y(t-h) \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix} h + A\begin{bmatrix} \int_{t-2h}^{t-h} z_1(s)ds \\ \int_{t-2h}^{t-h} y(s)ds \end{bmatrix} + \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix}\int_{t-2h}^{t-h} z_2(s)ds + \begin{bmatrix} \int_{t-2h}^{t-h} \varepsilon_2(s)ds \\ \int_{t-2h}^{t} \varepsilon_1(s)ds \end{bmatrix} \tag{A6}$$

and substituting in (A5) we obtain

$$\begin{bmatrix} \int_{t-h}^{t} z_1(s)ds \\ \int_{t-h}^{t} y(s)ds \end{bmatrix} = e^{Ah} \begin{bmatrix} \int_{t-2h}^{t-h} z_1(s)ds \\ \int_{t-2h}^{t-h} y(s)ds \end{bmatrix} + A^{-1}\left(e^{Ah} - I\right)\begin{bmatrix} c_2 \\ c_1 \end{bmatrix} h +$$

$$A^{-1}\Delta_h \int_{t-h}^{t} e^{A(t-s)} \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} z_2(s)ds + A^{-1}e^{Ah}\begin{bmatrix} 0 \\ \beta_2 \end{bmatrix}\int_{t-2h}^{t-h} z_2(s)ds - A^{-1}\begin{bmatrix} 0 \\ \beta_2 \end{bmatrix}\int_{t-h}^{t} z_2(s)ds + \tag{A7}$$

$$A^{-1}\Delta_h \int_{t-h}^{t} e^{A(t-s)} \begin{bmatrix} \varepsilon_1(s) \\ \varepsilon_2(s) \end{bmatrix} ds + A^{-1}e^{Ah}\begin{bmatrix} \int_{t-2h}^{t-h} \varepsilon_1(s) \\ \int_{t-2h}^{t} \varepsilon_2(s) \end{bmatrix} ds - A^{-1}\begin{bmatrix} \int_{t-h}^{t} \varepsilon_2(s) \\ \int_{t-h}^{t} \varepsilon_1(s) \end{bmatrix} ds$$

PART B

Notice that for $s \in (t-h,t)$

$$z_2(s) = c_3(s-(t-h)) + z_2(t-h) + \int_{t-h}^{s} \varepsilon_3(\theta)d\theta$$

and for $s \in (t-2h,t-h)$

$$z_2(s) = c_3(s-(t-2h)) + z_2(t-2h) + \int_{t-2h}^{s} \varepsilon_3(\theta)d\theta$$

substituting the last two expressions in

$$\frac{\beta_2}{a}\int_{t-h}^{t}\left\{e^{a(t-s)}-1\right\}z_2(s)ds + \frac{\beta_2}{a}\int_{t-2h}^{t-h}\left\{e^{ah}-e^{a(t-h-s)}\right\}z_2(s)ds =$$
$$-\frac{\beta_2}{a}\int_{t-h}^{t}\left\{c_3(s-(t-h)) + z_2(t-h) + \int_{t-h}^{s}\varepsilon_3(\theta)d\theta\right\}ds +$$
$$\frac{\beta_2}{a}e^{ah}\int_{t-2h}^{t-h}\left\{c_3(s-(t-2h)) + z_2(t-2h) + \int_{t-2h}^{s}\varepsilon_3(\theta)d\theta\right\}ds +$$
$$\frac{\beta_2}{a}\int_{t-h}^{t}e^{a(t-s)}\left\{c_3(s-(t-h)) + z_2(t-h) + \int_{t-h}^{s}\varepsilon_3(\theta)d\theta\right\}ds -$$
$$\frac{\beta_2}{a}\int_{t-2h}^{t-h}e^{a(t-h-s)}\left\{c_3(s-(t-2h)) + z_2(t-2h) + \int_{t-2h}^{s}\varepsilon_3(\theta)d\theta\right\}ds =$$
$$-\frac{\beta_2}{a}hz_2(t-h) + \frac{\beta_2}{a}he^{ah}z_2(t-2h) +$$
$$\frac{\beta_2}{a}z_2(t-h)\int_{t-h}^{t}e^{a(t-s)}ds - \frac{\beta_2}{a}z_2(t-2h)\int_{t-2h}^{t-h}e^{a(t-h-s)}ds$$
$$+c_3h\frac{\beta_2}{a}(t-h) - \frac{\beta_2}{a}c_3\int_{t-h}^{t}sds - \frac{\beta_2}{a}he^{ah}c_3(t-2h) + \frac{\beta_2}{a}e^{ah}c_3\int_{t-2h}^{t-h}s\,ds$$
$$+\frac{\beta_2}{a}c_3\int_{t-h}^{t}e^{a(t-s)}sds - \frac{\beta_2}{a}c_3\int_{t-2h}^{t-h}e^{a(t-h-s)}sds - \frac{\beta_2}{a}c_3h\int_{t-2h}^{t-h}e^{a(t-h-s)}ds +$$
$$\vartheta_t$$

where

$$\vartheta_t = -\frac{\beta_2}{a}\int_{t-h}^{t}\int_{s-h}^{s}\varepsilon_3(\theta)d\theta ds + \frac{\beta_2}{a}e^{ah}\int_{t-2h}^{t-h}\int_{s-2h}^{s-h}\varepsilon_3(\theta)d\theta ds$$
$$+\frac{\beta_2}{a}\int_{t-h}^{t}\int_{s-h}^{s}e^{a(t-s)}\varepsilon_3(\theta)d\theta ds - \frac{\beta_2}{a}\int_{t-2h}^{t-h}\int_{s-2h}^{s-h}e^{a(t-h-s)}\varepsilon_3(\theta)d\theta ds$$

Moreover notice that

$$\frac{\beta_2}{a}c_3\int_{t-h}^{t}e^{a(t-s)}sds - \frac{\beta_2}{a}c_3\int_{t-2h}^{t-h}e^{a(t-h-s)}sds = -\frac{\beta_2}{a}c_3\frac{1}{a}h\left\{1-e^{ah}\right\}$$

while

$$-\frac{\beta_2}{a}c_3h\int_{t-2h}^{t-h}e^{a(t-h-s)}ds = -\frac{\beta_2}{a}c_3h\left[\frac{1-e^{ah}}{-a}\right]$$

therefore

$$\frac{\beta_2}{a}\int_{t-h}^{t}\left\{e^{a(t-s)}-1\right\}z_2(s)ds+\frac{\beta_2}{a}\int_{t-2h}^{t-h}\left\{e^{ah}-e^{a(t-h-s)}\right\}z_2(s)ds=$$

$$-\frac{\beta_2}{a}hz_2(t-h)+\frac{\beta_2}{a}he^{ah}z_2(t-2h)+$$

$$\frac{\beta_2}{a}z_2(t-h)\left[\frac{1-e^{ah}}{-a}\right]-\frac{\beta_2}{a}z_2(t-2h)\left[\frac{1-e^{ah}}{-a}\right]$$

$$-\frac{1}{2}h^2c_3\frac{\beta_2}{a}+\frac{1}{2}h^2c_3\frac{\beta_2}{a}e^{ah}$$

$$-\frac{\beta_2}{a}c_3\frac{h}{a}\left\{1-e^{ah}\right\}+\frac{\beta_2}{a}\frac{1}{a}c_3h\left[1-e^{ah}\right]+\vartheta_t$$

also from

$$z_2(t-2h)=z_2(t-h)-c_3h-\int_{t-2h}^{t-h}\varepsilon_{3t}(s)ds$$

we finally obtain relation (11)

$$\frac{\beta_2}{a}\int_{t-h}^{t}\left\{e^{a(t-s)}-1\right\}z_2(s)ds+\frac{\beta_2}{a}\int_{t-2h}^{t-h}\left\{e^{ah}-e^{a(t-h-s)}\right\}z_2(s)ds=$$

$$z_2(t-h)\frac{\beta_2}{-a}h\left(1-e^{ah}\right)+\frac{1}{2}h^2c_3\frac{\beta_2}{-a}\left(1-e^{ah}\right)$$

$$+c_3h\frac{\beta_2}{-a}\left\{he^{ah}-\frac{1-e^{ah}}{-a}\right\}+\frac{\beta_2}{-a}\left\{he^{ah}-\frac{1-e^{ah}}{-a}\right\}\int_{t-2h}^{t-h}\varepsilon_{3t}(s)ds+\vartheta_t$$

PART C

$$\begin{bmatrix}\xi_{1t}(h)\\ \xi_{2t}(h)\end{bmatrix} = A^{-1}\int_{t-h}^{t}\left\{e^{A(t-s)} - I\right\}\begin{bmatrix}\varepsilon_1(s)\\ \varepsilon_2(s)\end{bmatrix}ds + A^{-1}\int_{t-2h}^{t-h}\left\{e^{Ah} - e^{A(t-h-s)}\right\}\begin{bmatrix}\varepsilon_2(s)\\ \varepsilon_1(s)\end{bmatrix}ds =$$

$$\frac{1}{\gamma_1 a}\begin{bmatrix}\alpha & 0\\ -\beta_1 & \gamma_1\end{bmatrix}\int_{t-h}^{t}\begin{bmatrix} e^{\gamma_1(t-s)} - 1 & 0\\ \beta_1\dfrac{e^{\gamma_1(t-s)} - e^{a(t-s)}}{\gamma_1 - a} & e^{a(t-s)} - 1\end{bmatrix}\begin{bmatrix}\varepsilon_2(s)\\ \varepsilon_1(s)\end{bmatrix}ds +$$

$$\frac{1}{\gamma_1 a}\begin{bmatrix}\alpha & 0\\ -\beta_1 & \gamma_1\end{bmatrix}\int_{t-2h}^{t-h}\begin{bmatrix} e^{\gamma_1 h} - e^{\gamma_1(t-h-s)} & 0\\ \beta_1\left(\dfrac{e^{\gamma_1 h} - e^{ah}}{\gamma_1 - a} - \dfrac{e^{\gamma_1(t-h-s)} - e^{a(t-h-s)}}{\gamma_1 - a}\right) & e^{ah} - e^{a(t-h-s)}\end{bmatrix}\begin{bmatrix}\varepsilon_2(s)\\ \varepsilon_1(s)\end{bmatrix}ds =$$

$$\int_{t-h}^{t}\begin{bmatrix}\dfrac{1}{-\gamma_1}\left(1 - e^{\gamma_1(t-s)}\right)\varepsilon_2(s)\\ \left\{\dfrac{\beta_1}{\gamma_1 a}\left(1 - e^{\gamma_1(t-s)}\right) + \dfrac{\beta_1}{a}\dfrac{e^{\gamma_1(t-s)} - e^{a(t-s)}}{\gamma_1 - a}\right\}\varepsilon_2(s) + \dfrac{1}{-a}\left(1 - e^{a(t-s)}\right)\varepsilon_1(s)\end{bmatrix} +$$

$$\int_{t-2h}^{t-h}\begin{bmatrix}\dfrac{1}{\gamma_1}\left(e^{\gamma_1 h} - e^{\gamma_1(t-h-s)}\right)\varepsilon_2(s)\\ \left\{-\dfrac{\beta_1}{\gamma_1 a}\left(e^{\gamma_1 h} - e^{\gamma_1(t-h-s)}\right) + \dfrac{\beta_1}{a}\left(\dfrac{e^{\gamma_1 h} - e^{ah}}{\gamma_1 - a} - \dfrac{e^{\gamma_1(t-h-s)} - e^{a(t-h-s)}}{\gamma_1 - a}\right)\right\}\varepsilon_2(s) + \dfrac{1}{a}\left(e^{ah} - e^{a(t-h-s)}\right)\varepsilon_1(s)\end{bmatrix}ds$$

therefore

$$\xi_{1t} = -\frac{1}{\gamma_1}\left\{\int_{t-h}^{t}\left(1 - e^{\gamma_1(t-s)}\right)\varepsilon_2(s)ds - \int_{t-2h}^{t-h}\left(e^{\gamma_1 h} - e^{\gamma_1(t-h-s)}\right)\varepsilon_2(s)ds\right\}$$

$$\xi_{2t} = \int_{t-h}^{t}\left\{\frac{\beta_1}{\gamma_1 a}\left(1 - e^{\gamma_1(t-s)}\right) + \frac{\beta_1}{a}\frac{e^{\gamma_1(t-s)} - e^{a(t-s)}}{\gamma_1 - a}\right\}\varepsilon_2(s) +$$

$$\int_{t-2h}^{t-h}\left\{-\frac{\beta_1}{\gamma_1 a}\left(e^{\gamma_1 h} - e^{\gamma_1(t-h-s)}\right) + \frac{\beta_1}{a}\left(\frac{e^{\gamma_1 h} - e^{ah}}{\gamma_1 - a} - \frac{e^{\gamma_1(t-h-s)} - e^{a(t-h-s)}}{\gamma_1 - a}\right)\right\}\varepsilon_2(s)$$

$$\int_{t-h}^{t}\frac{1}{-a}\left(1 - e^{a(t-s)}\right)\varepsilon_1(s) + \int_{t-2h}^{t-h}\frac{1}{a}\left(e^{ah} - e^{a(t-h-s)}\right)\varepsilon_1(s)$$

PART D

Initial Conditions

Integrating the DGP (1) in the interval $[0,h]$ we obtain

$$y(h)-y(0)=c_1h+\alpha\int_0^h y(s)ds+\beta_1\int_0^h z_1(s)ds+\beta_2\int_0^h z_2(s)ds+\int_0^h \varepsilon_1(s)ds \tag{D1}$$

$$z_1(h)-z_1(0)=c_2h+\gamma\int_0^h z_1(s)ds+\int_0^h \varepsilon_2(s)ds \tag{D2}$$

$$z_2(h)-z_2(0)=c_3h+\int_0^h \varepsilon_3(s)ds \tag{D3}$$

from the first two equations we obtain

$$\begin{bmatrix} y(h) \\ z_1(h) \end{bmatrix}=e^{\begin{bmatrix} a & \beta_1 \\ 0 & \gamma \end{bmatrix}h}\begin{bmatrix} y(0) \\ z_1(0) \end{bmatrix}+\int_0^h e^{A(h-s)}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}ds+\int_0^h e^{A(h-s)}\begin{bmatrix} \beta_2 \\ 0 \end{bmatrix}z_2(s)ds+\int_0^h e^{A(h-s)}\begin{bmatrix} \varepsilon_1(s) \\ \varepsilon_2(s) \end{bmatrix}ds \tag{D4}$$

but

$$\begin{bmatrix} a & \beta_1 \\ 0 & \gamma \end{bmatrix}^{-1}=\begin{bmatrix} \frac{1}{a} & -\beta_1\frac{1}{a\gamma} \\ 0 & \frac{1}{\gamma} \end{bmatrix}$$

and

$$e^{\begin{bmatrix} a & \beta_1 \\ 0 & \gamma \end{bmatrix}(h-s)}=\begin{bmatrix} e^{a(h-s)} & \beta_1\frac{e^{a(h-s)}-e^{\gamma(h-s)}}{a-\gamma} \\ 0 & e^{\gamma(h-s)} \end{bmatrix}$$

therefore from D4

$$\begin{bmatrix} y(h) \\ z_1(h) \end{bmatrix}=\begin{bmatrix} e^{a(h)} & \beta_1\frac{e^{a(h)}-e^{\gamma(h)}}{a-\gamma} \\ 0 & e^{\gamma(h)} \end{bmatrix}\begin{bmatrix} y(0) \\ z_1(0) \end{bmatrix}-A^{-1}\left(I-e^{Ah}\right)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}+\beta_2\begin{bmatrix} \int_0^h e^{a(h-s)}z_2(s)ds \\ 0 \end{bmatrix}+\int_0^h e^{A(h-s)}\begin{bmatrix} \varepsilon_1(s) \\ \varepsilon_2(s) \end{bmatrix}ds \tag{D5}$$

Moreover from (D1) and (D2) we obtain

$$\begin{bmatrix} y(h) \\ z_1(h) \end{bmatrix}=\begin{bmatrix} y(0) \\ z_1(0) \end{bmatrix}+\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}h+\begin{bmatrix} a & \beta_1 \\ 0 & \gamma \end{bmatrix}\begin{bmatrix} \int_0^h y(s) \\ \int_0^h z_1(s) \end{bmatrix}ds+\begin{bmatrix} \beta_2 \\ 0 \end{bmatrix}\int_0^h z_2(s)ds+\int_0^h\begin{bmatrix} \varepsilon_1(s) \\ \varepsilon_2(s) \end{bmatrix}ds \tag{D6}$$

solving D6 we obtain

$$\begin{bmatrix} \int_0^h y(s) \\ \int_0^h z_1(s) \end{bmatrix} ds = \begin{bmatrix} a & \beta_1 \\ 0 & \gamma \end{bmatrix}^{-1} \left\{ \begin{bmatrix} y(h) \\ z_1(h) \end{bmatrix} - \begin{bmatrix} y(0) \\ z_1(0) \end{bmatrix} - \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} h - \begin{bmatrix} \beta_2 \\ 0 \end{bmatrix} \int_0^h z_2(s) ds - \int_0^h \begin{bmatrix} \varepsilon_1(s) \\ \varepsilon_2(s) \end{bmatrix} ds \right\}$$

(D7)

Substituting D5 in D6 we obtain

$$\begin{bmatrix} \int_0^h y(s) \\ \int_0^h z_1(s) \end{bmatrix} ds =$$

$$\begin{bmatrix} \frac{1}{a} & -\beta_1 \frac{1}{a\gamma} \\ 0 & \frac{1}{\gamma} \end{bmatrix} \left\{ \begin{bmatrix} e^{a(h)} & \beta_1 \frac{e^{a(h)} - e^{\gamma(h)}}{a-\gamma} \\ 0 & e^{\gamma(h)} \end{bmatrix} \begin{bmatrix} y(0) \\ z_1(0) \end{bmatrix} - A^{-1}\left(I - e^{Ah}\right) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \beta_2 \begin{bmatrix} \int_0^h e^{a(h-s)} z_2(s) ds \\ 0 \end{bmatrix} + \right.$$

$$\left. \int_0^h e^{A(h-s)} \begin{bmatrix} \varepsilon_1(s) \\ \varepsilon_2(s) \end{bmatrix} ds - \begin{bmatrix} y(0) \\ z_1(0) \end{bmatrix} - \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} h - \begin{bmatrix} \beta_2 \\ 0 \end{bmatrix} \int_0^h z_2(s) ds - \int_0^h \begin{bmatrix} \varepsilon_1(s) \\ \varepsilon_2(s) \end{bmatrix} ds \right\}$$

(D8)

therefore from D8

$$\begin{bmatrix} \int_0^h y(s) \\ \int_0^h z_1(s) \end{bmatrix} ds =$$

$$\begin{bmatrix} \frac{1}{a} e^{a(h)} & \frac{1}{a}\beta_1 \frac{e^{a(h)} - e^{\gamma(h)}}{a-\gamma} - \beta_1 \frac{1}{a\gamma} e^{\gamma(h)} \\ 0 & \frac{1}{\gamma} e^{\gamma(h)} \end{bmatrix} \begin{bmatrix} y(0) \\ z_1(0) \end{bmatrix} -$$

$$\left(\begin{bmatrix} \frac{1}{a^2} & -\frac{\beta_1(\gamma + a)}{(a\gamma)^2} \\ 0 & \frac{1}{\gamma^2} \end{bmatrix} - \begin{bmatrix} \frac{1}{a^2} e^{a(h)} & \frac{\beta_1}{a^2} \frac{e^{a(h)} - e^{\gamma(h)}}{a-\gamma} - \beta_1 e^{\gamma(h)} \frac{a+\gamma}{a^2\gamma^2} \\ 0 & \frac{1}{\gamma^2} e^{\gamma(h)} \end{bmatrix} \right) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} +$$

$$\begin{bmatrix} \frac{\beta_2}{a} \int_0^h e^{a(h-s)} z_2(s) ds \\ 0 \end{bmatrix} + \int_0^h \begin{bmatrix} \frac{1}{a}\left(e^{a(h-s)} \varepsilon_1(s) ds + \beta_1 \frac{e^{a(h-s)} - e^{\gamma(h-s)}}{a-\gamma} \varepsilon_2(s) ds \right) - \beta_1 \frac{1}{a\gamma} e^{\gamma(h-s)} \varepsilon_2(s) ds \\ \frac{1}{\gamma} e^{\gamma(h-s)} \varepsilon_2(s) ds \end{bmatrix} -$$

$$\begin{bmatrix} \frac{1}{a} & -\beta_1 \frac{1}{a\gamma} \\ 0 & \frac{1}{\gamma} \end{bmatrix} \left(\begin{bmatrix} y(0) \\ z_1(0) \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} h \right) - \begin{bmatrix} \frac{\beta_2}{a} \int_0^h z_2(s) ds \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{a} & -\beta_1 \frac{1}{a\gamma} \\ 0 & \frac{1}{\gamma} \end{bmatrix} \int_0^h \begin{bmatrix} \varepsilon_1(s) \\ \varepsilon_2(s) \end{bmatrix} ds.$$

and finally

$$\int_0^h y(s) = \frac{1}{a}e^{a(h)}y(0) + \left(\frac{1}{a}\beta_1\frac{e^{a(h)} - e^{\gamma(h)}}{a-\gamma} - \beta_1\frac{1}{a\gamma}e^{\gamma(h)}\right)z_1(0) -$$

$$\left[\left(\frac{1}{a^2} - \frac{1}{a^2}e^{a(h)}\right)c_1 + \left(-\frac{\beta_1(\gamma+a)}{(a\gamma)^2} - \left(\frac{\beta_1}{a^2}\frac{e^{a(h)} - e^{\gamma(h)}}{a-\gamma} - \beta_1 e^{\gamma(h)}\frac{a+\gamma}{a^2\gamma^2}\right)\right)c_2\right] + \frac{\beta_2}{a}\int_0^h e^{a(h-s)}z_2(s)ds +$$

$$\int_0^h\left(\frac{1}{a}\left(e^{a(h-s)}\varepsilon_1(s)ds + \beta_1\frac{e^{a(h-s)} - e^{\gamma(h-s)}}{a-\gamma}\varepsilon_2(s)ds\right) - \beta_1\frac{1}{a\gamma}e^{\gamma(h-s)}\varepsilon_2(s)ds\right) -$$

$$\frac{1}{a}\left(y(0) + c_1 h\right) + \beta_1\frac{1}{a\gamma}\left(z_1(0) + c_2 h\right) - \frac{\beta_2}{a}\int_0^h z_2(s)ds - \frac{1}{a}\int_0^h \varepsilon_1(s)ds + \beta_1\frac{1}{a\gamma}\int_0^h \varepsilon_2(s)ds$$

but from $dz_2(t) = c_3 ds + \varepsilon_3(s)ds$

we obtain for $s \in (0,h)$

$$z_2(s) = c_3(s) + z_2(0) + \int_0^s \varepsilon_3(\theta)d\theta$$

then

$$\int_0^h e^{a(h-s)}z_2(s)ds = z_2(0)\frac{1-e^{ah}}{-a} + c_3\frac{1}{-a}\left[h - \frac{1-e^{ah}}{-a}\right] + \int_0^h\int_0^s e^{a(h-s)}\varepsilon_3(\theta)d\theta ds$$

also

$$\int_0^h z_2(s)ds = z_2(0)h + \frac{1}{2}c_3 h^2 + \int_0^h\int_0^s \varepsilon_3(\theta)d\theta ds$$

therefore

$$\int_0^h y(s) = \frac{1}{a}\left(e^{a(h)} - 1\right)y(0) + \left(\frac{1}{a}\beta_1\frac{e^{a(h)} - e^{\gamma(h)}}{a-\gamma} - \beta_1\frac{1}{a\gamma}\left(e^{\gamma(h)} - 1\right)\right)z_1(0) + z_2(0)\frac{\beta_2}{a}\left(\frac{1-e^{ah}}{-a} - h\right)$$

$$-\left[\frac{1}{a}\left(\frac{1}{a}\left(1 - e^{a(h)}\right) + h\right)c_1 + \left(-\frac{\beta_1}{a^2}\frac{e^{a(h)} - e^{\gamma(h)}}{a-\gamma} + \beta_1\left(e^{\gamma(h)} - 1\right)\frac{a+\gamma}{a^2\gamma^2} - \beta_1\frac{1}{a\gamma}h\right)c_2\right] +$$

$$+ c_3\frac{1}{-a}\left[h - \frac{1-e^{ah}}{-a}\right]\frac{\beta_2}{a} - \frac{\beta_2}{a}\frac{1}{2}c_3 h^2$$

$$+ \frac{1}{a}\int_0^h\left(e^{a(h-s)} - 1\right)\varepsilon_1(s)ds + \frac{1}{a}\beta_1\int_0^h\frac{e^{a(h-s)} - e^{\gamma(h-s)}}{a-\gamma}\varepsilon_2(s)ds - \beta_1\frac{1}{a\gamma}\int_0^h e^{\gamma(h-s)}\varepsilon_2(s)ds$$

$$+ \beta_1\frac{1}{a\gamma}\int_0^h \varepsilon_2(s)ds + \frac{\beta_2}{a}\int_0^h\int_0^s e^{a(h-\theta)}\varepsilon_3(\theta)d\theta ds - \frac{\beta_2}{a}\int_0^h\int_0^s \varepsilon_3(\theta)d\theta ds$$

but

$$E\left[\int_0^h\left(e^{a(h-s)} - 1\right)\varepsilon_1(s)ds\right]^2 = \sigma_{\varepsilon_1}^2\int_0^h\left(e^{a(h-s)} - 1\right)^2 ds = \sigma_{\varepsilon_1}^2\left[\frac{1}{-2a}e^{2a(h-s)}\Big|_{s=0}^{s=h} + h - \frac{2}{-a}e^{a(h-s)}\Big|_{s=0}^{s=h}\right] =$$

$$\sigma_{\varepsilon_1}^2\left[\frac{1}{-2a}\left(1 - e^{2a}\right) + h - \frac{2}{-a}\left(1 - e^{a}\right)\right]$$

therefore

$$\int_0^h \left(e^{a(h-s)} - 1\right)\varepsilon_1(s)ds \equiv \pi_1 \zeta_{\varepsilon_{1h}}$$

$$\pi_1 = \left[\frac{1}{-2a}\left(1-e^{2a}\right) + h - \frac{2}{-a}\left(1-e^{a}\right)\right]^{\frac{1}{2}}$$

$$\zeta_{\varepsilon_{1h}} \to N(0, \sigma^2_{\varepsilon_1})$$

and

$$\int_0^h \left\{\frac{e^{a(h-s)} - e^{\gamma(h-s)}}{a-\gamma} - \frac{1}{\gamma}e^{\gamma(h-s)} + \frac{1}{\gamma}\right\}\varepsilon_2(s)ds$$

$$= \pi_2 \zeta_{\varepsilon_{2h}}$$

$$\pi_2 = \left\{\int_0^h \left\{\frac{e^{a(h-s)} - e^{\gamma(h-s)}}{a-\gamma} - \frac{1}{\gamma}e^{\gamma(h-s)} + \frac{1}{\gamma}\right\}^2 ds\right\}^{\frac{1}{2}}$$

$$\zeta_{\varepsilon_{2h}} \to N(0, \sigma^2_{\varepsilon_2})$$

also

$$\int_0^h \int_0^s e^{a(h-\theta)}\varepsilon_3(\theta)d\theta ds = \int_0^h \left[\int_\theta^h e^{a(h-s)}ds\right]\varepsilon_3(\theta)d\theta + \int_{-h}^0 \left[\int_0^{\theta+h} e^{a(h-s)}ds\right]\varepsilon_3(\theta)d\theta =$$

$$\frac{1}{-a}\int_0^h \left(1-e^{a(h-\theta)}\right)\varepsilon_3(\theta)d\theta + \frac{1}{-a}\int_{-h}^0 \left(e^{-a\theta} - e^{ah}\right)\varepsilon_3(\theta)d\theta$$

moreover

$$\int_0^h \int_0^s \varepsilon_3(\theta)d\theta ds = \int_0^h \left[\int_\theta^h ds\right]\varepsilon_3(\theta)d\theta + \int_{-h}^0 \left[\int_0^{\theta+h} ds\right]\varepsilon_3(\theta)d\theta =$$

$$\int_0^h (h-\theta)\varepsilon_3(\theta)d\theta + \int_{-h}^0 (\theta+h)\varepsilon_3(\theta)d\theta$$

Now suppose that $\varepsilon_2(s) = \varepsilon_3(s) = 0, s \in (-h, 0)$ then:

$$\int_0^h \int_0^s e^{a(h-\theta)}\varepsilon_3(\theta)d\theta ds = \frac{1}{-a}\int_0^h \left(1-e^{a(h-\theta)}\right)\varepsilon_3(\theta)d\theta = \pi_3 \zeta_{\varepsilon_{3h}}$$

$$\pi_3 = \left(\frac{1}{a}\right)^2 \int_0^h \left(1-e^{a(h-\theta)}\right)^2 d\theta$$

$$\zeta_{\varepsilon_{3h}} \to N(0, \sigma^2_{\varepsilon_3})$$

and

$$\int_0^h \int_0^s \varepsilon_3(\theta) d\theta ds = \int_0^h (h-\theta) \varepsilon_3(\theta) d\theta = \pi_4 \zeta_{\varepsilon_{3h}}$$

$$\pi_4 = \int_0^h (h-\theta)^2 \, d\theta$$

finally

$$\int_0^h y(s) = \frac{1}{a}\left(e^{a(h)}-1\right)y(0) + \left(\frac{1}{a}\beta_1 \frac{e^{a(h)}-e^{\gamma(h)}}{a-\gamma} - \beta_1 \frac{1}{a\gamma}\left(e^{\gamma(h)}-1\right)\right) z_1(0) + z_2(0)\frac{\beta_2}{a}\left(\frac{1-e^{ah}}{-a}-h\right)$$

$$-\left[\frac{1}{a}\left(\frac{1}{a}\left(1-e^{a(h)}\right)+h\right)c_1 + \left(-\frac{\beta_1}{a^2}\frac{e^{a(h)}-e^{\gamma(h)}}{a-\gamma} + \beta_1\left(e^{\gamma(h)}-1\right)\frac{a+\gamma}{a^2\gamma^2} - \beta_1 \frac{1}{a\gamma} h\right)c_2\right] +$$

$$+ c_3 \frac{1}{-a}\left[h - \frac{1-e^{ah}}{-a}\right]\frac{\beta_2}{a} - \frac{\beta_2}{a}\frac{1}{2}c_3 h^2$$

$$+ \frac{1}{a}\pi_1 \zeta_{\varepsilon_{1h}} + \frac{\beta_1}{a}\pi_2 \zeta_{\varepsilon_{2h}} + \frac{\beta_2}{a}\pi_3 \zeta_{\varepsilon_{3h}} - \frac{\beta_2}{a}\pi_4 \zeta_{\varepsilon_{3h}}$$

(D9)

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